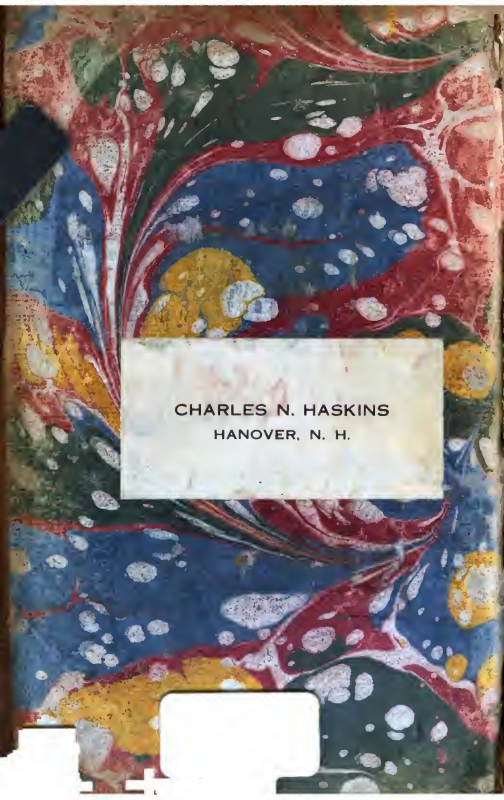




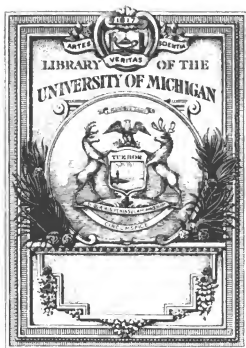
The image shows the front cover of a book. The cover is decorated with a traditional marbled paper pattern, featuring swirling veins of red, blue, green, and yellow, interspersed with white spots. A rectangular white label is pasted in the center of the cover. The label contains the text "CHARLES N. HASKINS" on the first line and "HANOVER, N. H." on the second line. At the bottom of the cover, there is a white, rectangular cutout or a piece of tape. The edges of the book's pages are visible on the left side.

CHARLES N. HASKINS
HANOVER, N. H.



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April 1895.



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THE
DOCTRINE
OF

ANNUITIES and REVERSIONS,

Deduced from

GENERAL and EVIDENT PRINCIPLES

WITH USEFUL

TABLES,

Shewing the VALUES of

SINGLE and JOINT LIVES, &c.

At different

RATES of INTEREST.

To which is added,

A METHOD of investigating the Value
of ANNUITIES by Approximation, without
the help of TABLES.

The Whole explain'd in a plain and simple Manner,
and illustrated by great variety of Examples.

By THOMAS SIMPSON.

LONDON:

Printed for J. NOURSE, at the *Lamb*, without *Temple-Bar*.
M.DCC.XLII.

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2. The second part of the paper is devoted to a discussion of the general principles of the theory of the structure of the atom.

3. The third part of the paper is devoted to a discussion of the general principles of the theory of the structure of the atom.

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THE
PREFACE.

THE subject upon which this little Tract is founded, as to its usefulness, needs no apology; and I have endeavoured to put it in such a light, as may fully answer the ends and expectation of the reader.

For, in the first place, I have given a very exact table for estimating the probability of life, deduced from 10 years observations on the bills of mortality of the city of London, whereupon the succeeding calculations are grounded. Then, after
A z shew-

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shewing how to compute the values of single lives, I lay down a Lemma, for the sake of those unacquainted with the principles of chances, by help whereof the most intricate Problems in the subject are resolved. Next the values of annuities upon more lives than one come to be considered; as, first, for any number of joint lives; secondly, for the longest of any number of lives; thirdly, for lives, where the annuity ceases upon the extinction of any assigned number of them. Then I proceed to determine the values of reversions, first, for the longest of any number of lives, after the longest of any number of other lives; secondly, for any number of joint lives, after any number of joint lives; thirdly, for any number of joint lives, after the longest of any number of lives; fourthly, for the longest of any number of lives, after any number of joint lives. Then, from the theorems before laid down, is given a set of tables for the valuation of annuities, upon one, two, or three lives, according to several rates of interest; the uses of which are explained in such a manner, as to be understood by all who know but common arithmetick.

Next

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Next is shewn how to determine the values of Successive Lives, where the first possessor has a right, at his decease, to nominate his successor, and his successor a next successor, and so on. Then the value that ought to be paid for renewing of leases upon any number of lives, together with the loss or gain of the purchaser in renewing for any assigned sum, is consider'd; as also how much the rent-roll of an estate ought to be increased upon account of such renewals.

Then is given a method for finding the values of Reversions, when the expectation depends on the chance of one particular life in possession surviving the rest.

Lastly, are laid down some easy practical Rules for approximating the values of lives without the help of tables.

*What, I apprehend, may best recommend this performance, is the general, yet familiar manner in which the subject is treated,
there*

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there not being a Solution throughout the whole work, except those relating to the use of the tables, that is not universal, according to any table of observations or degree of probability of life whatsoever; and yet the conclusions and practical rules deduced therefrom, are, for the most part, altogether as simple, as could be derived from any hypothesis. I mention hypotheses, because some authors on this subject, without troubling themselves or their readers about observations, &c. have taken upon them to prescribe methods of their own, that have neither foundation in experience nor in reason. But tho' I cannot help blaming those men, who would thus arbitrarily obtrude their own notions upon the world for infallible rules; yet I would not be thought to condemn any hypothesis grounded upon reason and matters of fact, because such are oftentimes made use of to very great advantage; of which Mr. De Moivre's excellent book on this subject is an instance.

Having taken some notice of the defects of others, it may not be improper to endeavour

THE PREFACE. vii

deavour to obviate an objection that may be made to the ensuing work.

It is possible that the great difference which there is, in one part of life, between the value of an annuity according to the Breslau observations, and the following tables, may tempt some (especially those whose interest it is) to question the exactness of the tables, or the observations whereon they were grounded; they may affirm that London consists of too flux a body to admit of any certain measure for the probability of life, and that the accounts published by the company of parish-clerks, are not to be depended on. But to this I answer, that tho' the continual resort of people from all parts, causes indeed a great increase in the bills of mortality, it will no ways influence the values of the annuities thence deduced, if the numbers of persons coming up to town at all ages, be proportional to the whole numbers of the living of the same ages; and tho' this supposition is not exactly true in small ages, yet as experience shews it to be more nearly so in greater, and as the number of persons that come to live in town after 25 or 30 years of age,

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age, is inconsiderable with respect to the whole body of inhabitants, it is evident, that the values given in the tables for all ages, not less than 25 or 30 years, can be but little effected from the cause abovementioned. 'Tis true, the values for younger lives, have not quite so good a foundation; but, I presume, the method I have had recourse to upon this occasion is such, as is not liable to any reasonable objections; and, as to the difference that may arise from any uncertainty or error in the accounts of the parish-clerks, it can be but very little, because if the age happens to be given in a little too high one time, there is the same chance of its being put down as much too low another.

Note. The Cut belonging to Lemma II. (being through Oversight there omitted) is inserted at the end of the Book.

O F

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the generality of people live to a greater age than at *London* (as appears by comparing the bills of mortality here, with those observations) can be no just measure of the probability of life in this place; and as to that of Mr. *Smart*, tho' it is indeed free from this objection, and founded on a very large number of observations, yet the great and continual afflux of people from all parts up to town, renders the deductions from those observations considerably different, in one part of life, from what they would otherwise be; and this Mr. *Smart* seems not, in his table, to have considered, or made any allowance for.

For these reasons, tho' I had determined to depend on, and make use of, this last gentleman's observations, in the ensuing pages (as, undoubtedly, the best for the city of *London*, and parts adjacent) yet have I deem'd it necessary to make some alterations, in the table of the probability of life, from thence derived.

In doing this, I have supposed the number of persons coming to live in town, after 25 years of age, to be inconsiderable, with respect to the whole number of inhabitants; and therefore the probabilities of life, for all ages above 25 years, the same as this author has made them; but then have increased the numbers of the living, corresponding to all ages below 25; so that they

Annuities *upon* Lives. 3

they may, as near as possible, be in the same proportion one to another, as they would be, were they to be deduced from observations on the mortality of those persons *only*, that are born within the bills. Which was done, by comparing together the number of christenings and burials, and observing, by help of Dr. *Halley's* table, the proportion which there is between the degrees of mortality at *London* and *Breslau*, in the other parts of life, where the ages are greater than 25. I shall here subjoin the table, altered as above, and then proceed immediately to the uses thereof,

A TABLE shewing the Proba-

Note. The Numbers mark'd x are supposed to die called the Decrements of Life.

N ^o . of Ages Persons. cur ^t .	N ^o . of Ages Persons. cur ^t .	N ^o of Ages Persons. cur ^t .	N ^o . of Ages Persons. cur ^t .
1280 born	524—10	462—20	385—30
410 x	7 x	7 x	9 x
870—1	517—11	455—21	376—31
170 x	7 x	7 x	9 x
700—2	510—12	448—22	367—32
65 x	6 x	7 x	9 x
635—3	504—13	441—23	358—33
35 x	6 x	7 x	9 x
600—4	498—14	434—24	349—34
20 x	6 x	8 x	9 x
580—5	492—15	426—25	340—35
16 x	6 x	8 x	9 x
564—6	486—16	418—26	331—36
13 x	6 x	8 x	9 x
551—7	480—17	410—27	322—37
10 x	6 x	8 x	9 x
541—8	474—18	402—28	313—38
9 x	6 x	8 x	9 x
532—9	468—19	394—29	304—39
8 x	6 x	9 x	10 x
524—10	462—20	385—30	294—40

bilities of LIFE, from Observations.

off yearly, and are what, in the succeeding Pages, are

N ^o . of Ages Persons. cur ^t .	N ^o . of Ages Persons. cur ^t .	N ^o . of Ages Persons. cur ^t .	N ^o . of Ages Persons. cur ^t .
294—40 10 x	204—50 8 x	130—60 7 x	69—70 5 x
284—41 10 x	196—51 8 x	123—61 6 x	64—71 5 x
274—42 10 x	188—52 8 x	117—62 6 x	59—72 5 x
264—43 9 x	180—53 8 x	111—63 6 x	54—73 5 x
255—44 9 x	172—54 7 x	105—64 6 x	49—74 4 x
246—45 9 x	165—55 7 x	99—65 6 x	45—75 4 x
237—46 9 x	158—56 7 x	93—66 6 x	41—76 3 x
228—47 8 x	151—57 7 x	87—67 6 x	38—77 3 x
220—48 8 x	144—58 7 x	81—68 6 x	35—78 3 x
212—49 8 x	137—59 7 x	75—69 6 x	32—79 3 x
204—50	130—60	69—70	29—80

Now,

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Now, in order to shew the use of the foregoing table by an example, let it be required to find the probability, that a person of thirty-six, lives 30 years longer, or attains to the age of 66 years, look in the table against 36 years and 66 years, and corresponding thereto, you will find the numbers 331 and 93 respectively; shewing, that out of 331 persons living of 36 years of age, only 93 of them arrive to the age of 66: therefore, seeing the whole number of persons living at the beginning of this term, is to the number remaining alive at the end of it, in the ratio of 331 to 93; the number of chances that a person of 36 years of age has to live 30 years longer, will be to the number of all the chances, that he has both to live beyond, and die within 30 years, in the same ratio of 331 to 93; and therefore $\frac{93}{331}$ is the measure of the probability required; the probability of the happening of any event, being always to be considered as the ratio of the chances which that event has to happen, to all the chances which it has both to happen and fail.

This being understood, suppose it were now required to find the value of an annuity of 100*l.* for a life of 20 years of age, interest at 4 *per cent.* .

Because

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Because the present value of 100*l.* due at the end of one year (discount being allowed) is 96.15, it is plain, that so much would be the value of the first year's rent, was the purchaser sure to receive it; but the probability of his living one year, appearing from the table to be only $\frac{455}{462}$, the aforesaid sum 96.15, in order to make a just deduction out of it, for the contingency of his dying before the end of one year, ought to be diminish'd in the ratio of 462 to 455, or multiply'd by $\frac{455}{462}$, which will reduce it to 94.70, equal to the true value of the first year's rent. After the same manner may the value of the second year's rent be calculated; for since the probability of receiving this rent, or living two years is $\frac{448}{462}$, let this be multiplied into 92.45, the present value of 100*l.* to be received at the end of two years, and the product 89.65, will be the true value of the second year's rent.

And by a like way of proceeding, the values of the 3d, 4th, 5th, &c. year's rents, to the utmost extent of life, may be determined; and the sum of all these will be the required value of the annuity; which will be found to come out 1480*l.* very near.

From

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From the same method of proceeding, the value of an annuity for any other life may be determined; and tho' the operations, requisite to this effect, are very numerous, yet, that being once computed, the value of the next younger life may from thence be easily derived; for if to the given value you add one year's purchase, and multiply the sum, discounted for one year, by the probability of the youngest life continuing one year, the product will be the value required (as will appear from Corollary VII. of the succeeding Problem.) Suppose, for example, that the value of a life of nineteen were required from the value of a life of twenty, as above computed; then, the value of the given life, increased by one year's purchase, being 1580*l.* the same discounted for one year, at 4 *per cent.* will be 1519.2, which multiply'd by $\frac{462}{468}$, the probability of a life of nineteen continuing one year, gives 1499.8, for the required value of an annuity upon this life.

Having shewn the manner of estimating the value of an annuity for any single life, and laid down a ready method of computing tables for such lives, according to any proposed rate of interest, by deducing each value from that of the next older life, it remains next to consider the manner of deter-

determining the values of annuities granted upon two or more lives; but before we enter upon this, it will be necessary (for the benefit of those unacquainted with the principles of chances) to premise the following.

L E M M A I.

If a sum of money (S) depends upon two different events, so as to be received, if both those events happen; the probability of receiving that sum, will be equal to the product of the probabilities of the happening of the two events.

Let the number of chances for the happening of one of the events, be to the number of all the chances, both for its happening and failing, as a to b ; and let the number of chances, for the happening of the other event, be to the number of all the chances, both for its happening and failing, as c to d ; then it is manifest, that $\frac{c}{d} \times S$, or the given sum diminished in the ratio of d to c , would be the true value of the expectation upon this sum (considered without regard to time) was the same to be received, or to depend only, upon the happening of the last named of

C

the

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the two proposed events; or, in other
Words, $\frac{c}{d} \times S$, will be the sum that might
be taken as an equivalent in this case, for
the chance of obtaining the sum S . But
the probability of the happening of the first
event, or of being intitled to receive the
sum $\frac{c}{d} \times S$, as an equivalent being only
 $\frac{a}{b}$, the value of the expectation; there-
fore, as it depends on both events, can be
only the $\frac{a}{b}$ part of this sum, or $\frac{a}{b} \times \frac{c}{d}$
 $\times S$, and consequently the probability of
receiving the sum S , only $\frac{a}{b} \times \frac{c}{d}$, *which*
was to be proved.

COROLLARY I.

Hence, also, may the probability of receiv-
ing any sum of money, depending on the
happening of 3, 4, &c. events be easily de-
rived: For let $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, $\frac{g}{h}$, &c. repre-
sent the probabilities of happening of so
many different events; and S , T , U , W ,
&c. several sums depending thereon, the
first S , to be received upon the happening
of the two first; the second T , upon the
happening of the three first; the third U ,
upon the happening of the four first, &c.
reckon-

Annuities upon Lives. 11

reckoning according to the order in which the respective probabilities are placed.

Then, seeing the probability of receiving the first sum S , is $\frac{a}{b} \times \frac{c}{d}$, and that the expectation on the second T , depends entirely upon the happening of this event, and that of the aforesaid order, whose probability is $\frac{e}{f}$, it follows that $\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f}$, the product of those two, will be the probability of receiving the sum T . In like manner, as the expectation on the sum U , depends upon the receiving of T , and the happening of the event, whose probability is $\frac{g}{h}$, the probability of receiving the sum U , will be $\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} \times \frac{g}{h}$, &c. whence it appears, that the probability of the happening of any number of events, is equal to the product of all the probabilities of happening of those events, considered separately.

P R O B L E M I.

The probability of life, and rate of interest being given; to find the value of an annuity, granted upon any number of joint lives, that is, for as long as they shall all continue in being together.

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SOLUTION.

Let the proposed annuity be 1 L, and A, B, C, &c. the lives upon which it is granted, and r the amount of 1 L, in one year, *viz.* principal and interest; let the probability of the life A continuing 1, 2, 3, &c. years, be represented by $a, a', a'', a''',$ &c. respectively; and that of the life B continuing 1, 2, 3, &c. years, by $b, b', b'', b''',$ &c. &c. then will the probability of all the lives continuing, 1, 2, 3, &c. years, be $abcd, \&c. a'b'c'd', \&c. a''b''c''d'', \&c. \&c.$ &c. respectively, by the Corollary to the preceeding Lemma. These, therefore, being respectively multiply'd into the terms of the geometric progression, $\frac{1}{r}, \frac{1}{r^2}, \frac{1}{r^3},$ &c. shewing the present value of 1 L certain, to be received at the expiration of 1, 2, 3, &c. years, the products $\frac{abcd, \&c.}{r}, \frac{a'b'c'd', \&c.}{r^2},$
 $\frac{a''b''c''d'', \&c.}{r^3},$ &c. thence arising, will respectively express the present values of the 1st, 2d, 3d, &c. year's rents, upon the contingency of one or more of the lives failing, in 1, 2, 3, &c. years: The sum of all which, or $\frac{abcd, \&c.}{r} + \frac{a'b'c'd', \&c.}{r^2} + \frac{a''b''c''d'', \&c.}{r^3}$
 &c. is, therefore, the present value of the annuity, Q. E. I. C Q.

COROLLARY I.

Hence, will the value of an annuity for one single life A, be express'd by $\frac{a}{r} + \frac{a'}{r^2} + \frac{a''}{r^3} + \frac{a'''}{r^4}$, &c. for two joint lives, A B, by $\frac{ab}{r} + \frac{ab'}{r^2} + \frac{ab''}{r^3} + \frac{ab'''}{r^4}$, &c. and for three joint lives by $\frac{abc}{r} + \frac{abc'}{r^2} + \frac{abc''}{r^3} + \frac{abc'''}{r^4}$, &c.

COROLLARY II. This part is written in the margin.

If a, a', a'', a''' , &c. be taken equal to a, a^2, a^3, a^4 , &c. and b, b', b'', b''' , &c. equal to b, b^2, b^3, b^4 , &c. then will the value of the annuity be defined by the geometric progression $\frac{abcd, \&c.}{r} + \frac{a^2b^2c^2d^2, \&c.}{r^2} + \frac{a^3b^3c^3d^3, \&c.}{r^3} \div \&c.$ or its equal $\frac{abcd, \&c.}{r - abcd, \&c.}$

COROLLARY III.

Therefore, if the value of an annuity for each of the single lives be given, equal to M, N, P, Q, &c. respectively; then, accord-

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according to the above hypothesis, $\frac{a}{r-a}$,
being $= M$, $\frac{b}{r-b} = N$, $\frac{c}{r-c} = P$, &c.

we shall have $a = \frac{rM}{M+1}$, $b = \frac{rN}{N+1}$, $c = \frac{rP}{P+1}$, &c. and consequently the value of
an annuity for all the joint lives equal to

$\frac{rM \times rN \times rP \times rQ, \&c.}{r \times M+1 \times N+1 \times P+1, \&c. - rM \times rN \times rP, \&c.}$
upon a supposition that the probabilities of
continuing in being 1, 2, 3, &c. years,
are to one another, as the terms of a geo-
metric progression, or that the decrements
of life are in a constant ratio.

COROLLARY IV.

But if the probability of living an
assigned part of time, be supposed to de-
crease continually, to the extremity of old
age, so that the terms of the series a ,
 $\overset{1}{a}$, $\overset{2}{a}$, $\overset{3}{a}$, &c. may be respectively expound-
ed by those of an arithmetic progression,
 $\frac{m-1}{m}$, $\frac{m-2}{m}$, $\frac{m-3}{m}$, $\frac{m-4}{m}$, &c. and the terms
of the series b , $\overset{1}{b}$, $\overset{2}{b}$, $\overset{3}{b}$, &c. by those of an
arithmetic progression, $\frac{n-1}{n}$, $\frac{n-2}{n}$, $\frac{n-3}{n}$, $\frac{n-4}{n}$,
&c. &c. then will the value of an annuity,
upon all the joint lives, it is manifest, be

$\frac{m-1}{r^m} \times \frac{n-1}{n} \times \frac{p-1, \&c.}{p} + \frac{m-2}{r^{2m}} \times \frac{n-2, \&c.}{n} +$
 $\frac{m-3}{r^3 m} \times \frac{n-3, \&c.}{n} \&c.$ where $m, n, p, \&c.$ respectively represent the numbers of years, which the several lives have a chance to continue, reckoning to the extremity of old age.

COROLLARY V.

Hence it appears that the value of an annuity, according to this last hypothesis, for one single life A will be $= \frac{m-1}{r^m} + \frac{m-2}{r^{2m}} + \frac{m-3}{r^3 m} + \frac{m-4}{r^4 m}, \&c.$ for two joint lives A B, equal to $\frac{m-1 \times n-1}{r^m n} + \frac{m-2 \times n-2}{r^{2m} n} + \frac{m-3 \times n-3}{r^3 m n}, \&c.$ and for three joint lives, A, B, C, equal to $\frac{m-1 \times n-1 \times p-1}{r^m n p} + \frac{m-2 \times n-2 \times p-2}{r^{2m} n p} + \frac{m-3 \times n-3 \times p-3}{r^3 m n p}, \&c.$ each series being to be continued to a number of terms (m), equal to the number of years included between the oldest life A, and the extremity of old age.

But these series may be summed, and will be found equal to $\frac{1}{r-1} - \frac{r+r^{1-m}}{m \times r-1}, \frac{1}{r-1}$
 $- \frac{m+n-1 \times r + n-m-1 \times r^{1-m}}{m n \times r-1^2} + \frac{2r-2r^{1-m}}{m n \times r-1},$
 and

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$$\text{and } \frac{1}{r-1} = \frac{mnp - m - 1 \times n - 1 \times p - 1 \times r}{mnp \times r - 1^2} \\ + \frac{n - m - 1 \times p - m - 1 \times r^{1-m}}{mnp \times r - 1^2} + \frac{2m + 2n + 2p -}{mnp \times r - 1^3} \\ \frac{6 \times r + 4m - 2n - 2p + 6 \times r^{1-m}}{mnp \times r - 1^3} +$$

$$\frac{6r^{1-m} - 6r}{mnp \times r - 1^4}, \text{ respectively; which values, if}$$

M be put for the value $\left(\frac{1-r^{-m}}{r-1}\right)$ of an annuity, *certain*, for m years, and v for $\left(\frac{1}{r-1}\right)$ that of the same annuity for ever, will become $v = \frac{v+1 \times M}{m}$,

$$v = \frac{v+1}{n} \times n - m - 2v - 1 \times \frac{M}{m} + 2v,$$

$$\text{and } v = \frac{v+1 \times v}{np} \times 2n + 2p - m - 6v - 3$$

$$+ \frac{v+1}{np} \times n - m - 1 \times p - m - 1 \times \frac{M}{m}$$

$$- \frac{v+1 \times 2v}{pn} \times n + p - 2m - 3v - 3 \times \frac{M}{m},$$

respectively; shewing the worth of an annuity for one, two, or three joint lives; upon supposition, that the probabilities of living 1, 2, 3, &c. years, are to one another, as the terms of an arithmetic progression; or that the decrements of life, from year to year, are all equal one to another.

COROLLARY VI.

But, let the probability of life be what it will, the required value may be always determined by help of a table of observations, and the general expression foregoing; for let the number of the living, corresponding to the age of A, in the table, be represented by Q, and those answering to the next succeeding ages, in the table, by $\overset{\circ}{Q}$, $\overset{''}{Q}$, &c. respectively; and, in like manner, let the number of the living, answering to the age of B, be represented by R, and those answering to the next succeeding ages, by $\overset{\circ}{R}$, $\overset{''}{R}$, $\overset{'''}{R}$, &c. &c. &c. then the probability, that the life A continues 1, 2, 3, &c. years, being $\frac{\overset{\circ}{Q}}{Q}$, $\frac{\overset{''}{Q}}{Q}$, $\frac{\overset{'''}{Q}}{Q}$, &c. and that of the life B, continuing 1, 2, 3, &c. years, equal to $\frac{\overset{\circ}{R}}{R}$, $\frac{\overset{''}{R}}{R}$, $\frac{\overset{'''}{R}}{R}$, &c. &c. we shall, by substituting these several values, instead of a , a' , a'' , &c. b , b' , b'' , &c. in the general expression, have $\frac{\overset{\circ}{Q}\overset{\circ}{R}\overset{\circ}{S}, \&c.}{r\overset{\circ}{Q}\overset{\circ}{R}\overset{\circ}{S}, \&c.} + \frac{\overset{\circ}{Q}\overset{''}{R}\overset{''}{S}, \&c.}{r^2\overset{\circ}{Q}\overset{''}{R}\overset{''}{S}, \&c.} + \frac{\overset{''}{Q}\overset{''}{R}\overset{''}{S}, \&c.}{r^3\overset{''}{Q}\overset{''}{R}\overset{''}{S}, \&c.}, \&c.$ or $\frac{1}{\overset{\circ}{Q}\overset{\circ}{R}\overset{\circ}{S}, \&c.} \times \frac{\overset{\circ}{Q}\overset{\circ}{R}\overset{\circ}{S}, \&c.}{r} + \frac{\overset{''}{Q}\overset{''}{R}\overset{''}{S}, \&c.}{r^2} + \frac{\overset{'''}{Q}\overset{'''}{R}\overset{'''}{S}, \&c.}{r^3}, \&c.$ equal to the value of the annuity.

D

CO-

COROLLARY VII.

Hence, if the value (P) of the joint lives A, B, C, be given, or once computed, the value (K) of the next younger lives, $\overset{A}{A}, \overset{B}{B}, \overset{C}{C}, \&c.$ whose ages are, each, respectively, one year less than those of A, B, C, $\&c.$ may be easily derived; for let $\overset{Q}{Q}, \overset{R}{R}, \overset{S}{S}, \&c.$ be the numbers found in the table of observations, against those next younger ages; then, for the very same reasons that $\frac{\overset{Q}{Q} \overset{R}{R} \overset{S}{S}, \&c.}{\overset{r}{r} \overset{Q}{Q} \overset{R}{R} \overset{S}{S}, \&c.} + \frac{\overset{Q}{Q} \overset{R}{R} \overset{S}{S}, \&c.}{\overset{r^2}{r^2} \overset{Q}{Q} \overset{R}{R} \overset{S}{S}, \&c.}, \&c. \text{ is } =$
P, shall $\frac{\overset{Q}{Q} \overset{R}{R} \overset{S}{S}, \&c.}{\overset{r}{r} \overset{Q}{Q} \overset{R}{R} \overset{S}{S}, \&c.} + \frac{\overset{Q}{Q} \overset{R}{R} \overset{S}{S}, \&c.}{\overset{r^2}{r^2} \overset{Q}{Q} \overset{R}{R} \overset{S}{S}, \&c.}, \&c. \text{ be } =$
K: Wherefore, multiplying the former equation by $\overset{Q}{Q} \overset{R}{R} \overset{S}{S}, \&c.$ and the latter by $\overset{r}{r} \overset{Q}{Q} \overset{R}{R} \overset{S}{S}, \&c.$ and, taking one from the other, we have $\overset{Q}{Q} \overset{R}{R} \overset{S}{S}, \&c. = \overset{r}{r} K \overset{Q}{Q} \overset{R}{R} \overset{S}{S}, \&c.$
 $- P \overset{Q}{Q} \overset{R}{R} \overset{S}{S}, \&c.$ and consequently $K =$
 $\frac{\overset{1}{1} + P \times \overset{Q}{Q} \overset{R}{R} \overset{S}{S}, \&c.}{\overset{r}{r} \overset{Q}{Q} \overset{R}{R} \overset{S}{S}, \&c.} = \frac{1}{1 + P} \times \frac{\overset{Q}{Q} \overset{R}{R} \overset{S}{S}, \&c.}{\overset{r}{r} \overset{Q}{Q} \overset{R}{R} \overset{S}{S}, \&c.}$

COROLLARY VIII.

Lastly, from Coroll. IV, V, and the given value (P) of the joint lives A, B, C, $\&c.$ the value of an annuity upon an equal
number

number of other joint lives $\dot{A}$, $\dot{B}$, $\dot{C}$, &c. respectively, younger than the former, by any number of years, during which, the decrements of life may be esteemed equal, may be readily determined. Let s be the proposed number of years, or the common difference between the ages of A and $\dot{A}$, B and $\dot{B}$, C and $\dot{C}$, &c. and w ($= \frac{1}{r}$) the present value of 1 L , due at the end of s years; let the number answering to each of the ages A , B , &c. be taken from the table of observations, and divided by the preceding decrement, and let the quotients (m, n , &c.) be considered as the complements of those ages, to the extremity of old age, and let (S) the value of an annuity, answering to those complements, by Coroll. V. be accordingly found.

Moreover, having added s to each of the said quotients, and taken the sums ($m+s$, $n+s$, &c.) thence arising, as the complements of the ages, $\dot{A}$, $\dot{B}$, $\dot{C}$, &c. to the extremity of old age, let (T) the value of an annuity answering to these complements, in the same manner, be also found. Then it will follow, from what has been laid down in the forementioned Corollaries, that

$$T + \overline{P-S} \times \frac{w m n, \&c.}{m+s \times n+s, \&c.} \text{ will express the value}$$

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value of an annuity upon the joint lives,
A, B, C, &c.

These two last Corollaries will be found very useful, in computing tables for the valuation of annuities upon one single life, or 2, 3, or more *joint* lives, *as deducible from real observations*; and I have insisted more largely on this proposition, because the most intricate questions in the subject, may be referred to it, and readily solved by help of tables so computed; as, in the succeeding propositions, will be made to appear.

P R O B L E M II.

The same things being given as in the last Proposition; to find the value of an annuity, granted upon any number of assigned lives, that is, to continue as long as the longest of them is in being.

S O L U T I O N.

Let every thing be supposed as in the preceding problem; then since the probability that the life A, B, or C, &c. fails the first year, is express'd by $1-a$, $1-b$, or $1-c$, &c. respectively, the probability of all the lives, A, B, C, &c. failing the first year,

year, will be express'd by $\frac{1}{1-a} \times \frac{1}{1-b} \times \frac{1}{1-c}$, &c. (per Lemma) therefore that of some one or more of them, surviving the first year, will be $1 - \frac{1}{1-a} \times \frac{1}{1-b} \times \frac{1}{1-c}$, &c. In like manner, it will appear, that the probability of some one or more of the lives, surviving the second year, will be $1 - \frac{1}{1-a} \times \frac{1}{1-b}$, &c. &c. Therefore, if these several probabilities be respectively multiply'd into the terms of the geometric series $\frac{1}{r}$, $\frac{1}{r^2}$, $\frac{1}{r^3}$, &c. expressing the present value of 1 L certain, to be received at the end of 1, 2, 3, &c. years, the products thence arising, will respectively exhibit the present values of the 1st, 2d, 3d, &c. year's rents, upon the contingency of some one or more of the lives surviving the 1st, 2d, 3d, &c. years; the sum of all which products, or $\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4} + \frac{1}{r^5}$, &c.

$$= \frac{\frac{1-a}{r} \times \frac{1-b}{r} \times \frac{1-c}{r} \times \frac{1-d}{r} \times \frac{1-e}{r} \times \frac{1-f}{r} \times \frac{1-g}{r} \times \frac{1-h}{r} \times \frac{1-i}{r} \times \frac{1-k}{r} \times \frac{1-l}{r} \times \frac{1-m}{r} \times \frac{1-n}{r} \times \frac{1-o}{r} \times \frac{1-p}{r} \times \frac{1-q}{r} \times \frac{1-r}{r} \times \frac{1-s}{r} \times \frac{1-t}{r} \times \frac{1-u}{r} \times \frac{1-v}{r} \times \frac{1-w}{r} \times \frac{1-x}{r} \times \frac{1-y}{r} \times \frac{1-z}{r} \times \frac{1}{r}}{r^2} = \frac{1-a \times 1-b \times 1-c \times 1-d \times 1-e \times 1-f \times 1-g \times 1-h \times 1-i \times 1-k \times 1-l \times 1-m \times 1-n \times 1-o \times 1-p \times 1-q \times 1-r \times 1-s \times 1-t \times 1-u \times 1-v \times 1-w \times 1-x \times 1-y \times 1-z}{r^2}$$

is therefore the value of the annuity. Q. E. I.

COROLLARY I.

Hence, if the lives be all equal, and their number be represented by s , then abc , $\textcircled{c}$. becoming equal to each other, the value of the annuity will be expreffible by $\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}$, $\textcircled{c}$.

$$\frac{\overbrace{1-a}^I}{r} - \frac{\overbrace{1-a}^I}{r^2} + \frac{\overbrace{1-a}^I}{r^3} - \frac{\overbrace{1-a}^I}{r^4} + \frac{\overbrace{1-a}^I}{r^5}, \textcircled{c}.$$

COROLLARY II.

Since the value of the annuity, converted to fimple terms, is

$$\begin{array}{l} \frac{a}{r} + \frac{b}{r}, \textcircled{c}. - \frac{ab}{r} - \frac{ac}{r}, \textcircled{c}. + \frac{abc}{r} + \frac{abd}{r} \textcircled{c}. \\ \frac{a}{r^2} + \frac{b}{r^2}, \textcircled{c}. - \frac{ab}{r^2} - \frac{ac}{r^2}, \textcircled{c}. + \frac{abc}{r^2} + \frac{abd}{r^2} \textcircled{c}. \\ \frac{a}{r^3} + \frac{b}{r^3}, \textcircled{c}. - \frac{ab}{r^3} - \frac{ac}{r^3}, \textcircled{c}. + \frac{abc}{r^3} + \frac{abd}{r^3} \textcircled{c}. \\ \textcircled{c}. \textcircled{c}. \quad \textcircled{c}. \textcircled{c}. \quad \textcircled{c}. \textcircled{c}. \end{array}$$

where, according to the last proposition, the first collateral column expreffes the value of an annuity for the fingle life A; the fecond, the like for the fingle life B; the third and fourth, for the joint lives AB and AC, $\textcircled{c}$. $\textcircled{c}$. it follows, that the value of an annuity, to continue as long

long as any one of the lives A, B, C, D, &c. is in being, is equal to the sum of the values of all the *single* lives, *less* the values of all the *joint* lives, combined two and two, *more* the values of all the *joint* lives, combined three and three, *less* the values of all the *joint* lives, combined four and four, and so on. Therefore, when the values of the *joint* lives are given, the value of an annuity upon the longest life, will from hence, be likewise given.

COROLLARY III.

But when the lives are all equal, the values of every 2, or 3, &c. *joint* lives, will likewise be equal; therefore, if the value of each *single* life be represented by H, that of each two *joint* lives by $\overset{1}{H}$, that of each three *joint* lives by $\overset{2}{H}$, &c. the values of all the *single* lives, being s in number, will be $= s H$, and the values of all the *joint* lives, combined two and two, $= s \times \frac{s-1}{2} \overset{1}{H}$, &c. Whence it is manifest, that the value of the longest life, in this case, will be $s H - \frac{s}{4} \times \frac{s-1}{2} \overset{1}{H} + \frac{s}{4} \times \frac{s-1}{2} \times \frac{s-2}{3} \overset{2}{H} - \frac{s}{1} \times \frac{s-1}{2} \times \frac{s-2}{3} \times \frac{s-3}{4} \overset{3}{H} + \frac{s}{4} \times \frac{s-1}{2} \times \frac{s-2}{3} \times \frac{s-3}{4} \times \frac{s-4}{5} \overset{4}{H}$, &c.

COROLLARY IV.

If the probabilities of continuing in being 1, 2, 3, &c. years, be expounded by the terms of an arithmetic progression; or the decrements of life from year to year, be supposed equal (as in Coroll. IV. and V. of the last Proposition) and if mnp , &c. be respectively put for the numbers of years between the several ages A, B, C, &c. and the extremity of old age; and MNP , &c. be taken to represent the present values of an annuity certain for those numbers of years, and v that of the annuity for ever; then will the value of one life A be equal to $v - \frac{v+1}{m} \times \frac{M}{m}$, of two lives A B,

equal to $v - \frac{v+1}{n} \times \frac{M+N-2v+2v+1}{2v+1} \times \frac{M}{m}$

of three lives A, B, C, $= v - \frac{v+1}{np} \times$

$\frac{m+1^2 + 2m + 3v + 3 \times 2v \times \frac{M}{m} - \frac{v+1}{p} \times$

$\frac{n+2v+1 \times \frac{N}{n} + P + \frac{v+1 \times v}{np} \times$

$2n+m+6v+3$; &c. which values,

therefore, when the lives A, B, &c. are all equal, will become $v - \frac{v+1}{m} \times \frac{M}{m}$, $v -$

$\frac{v+1}{m} \times \frac{2m+2v+1}{m} \times \frac{M}{m} - 2v$, and $v -$

$\frac{v+1}{m^2} \times \frac{m+1 \times 3m+6v+6v^2+1}{m} \times \frac{M}{m}$

$\frac{3v \times v+1 \times m+2v+1}{m^2}$, respectively.

Note:

Note. When the proposed lives are unequal, or of different ages, A is to be taken as the oldest of them, B as the next oldest, and so on.

PROBLEM III.

To find the value of an annuity granted upon any number, n , of lives, A, B, C, &c. but so as to continue only as long as a given number, m , of them are in being.

SOLUTION.

Let P be the value of all the *joint* lives, A, B, C, &c. that is, the value of an annuity, for as long as they shall all continue in being together; and let Q be the sum of the values of all the *joint* lives that can arise, by combining A, B, C, &c. so as to leave out one life at each combination, and R, the sum of the values of all the *joint* lives, that can arise by combining the same, so as to leave out two lives at each combination, &c. &c. Then will

R

P x

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$$P \times 1 - \frac{n}{1} + n \times \frac{n-1}{2} - n \times \frac{n-1}{2} \times \frac{n-2}{3}, \&c.$$

$$Q \times 1 - \frac{n-1}{1} + \frac{n-1}{1} \times \frac{n-2}{2} - \frac{n-1}{1} \times \frac{n-2}{2} \times \frac{n-3}{3}$$

$$R \times 1 - \frac{n-2}{1} + \frac{n-2}{1} \times \frac{n-3}{2} - \frac{n-2}{1} \times \frac{n-3}{2} \times \frac{n-4}{3}$$

$$S \times 1 - \frac{n-3}{1} + \frac{n-3}{1} \times \frac{n-4}{2} - \frac{n-3}{1} \times \frac{n-4}{2} \times \frac{n-5}{3}$$

&c.

be the value of the annuity required; where the first series is to be continued to as many terms, as there are units in $n+1-m$; the second to as many terms, as the first all but one; the third to as many as the second, all but one, and so on.

COROLLARY.

Hence it appears, that the value of an annuity, to cease upon the failing of the first life, will be P ; upon the failing of the second life, $1-n \times P+Q$; upon failing of the third life, $1-n+n \times \frac{n-1}{2} \times P+2-nQ+R$; and upon failing of the fourth life, $1-n+n \times \frac{n-1}{2} - n \times \frac{n-1}{2} \times \frac{n-2}{3} \times P+2-n+\frac{n-1}{1} \times \frac{n-2}{2} \times Q+3-n \times R+S, \&c. \&c.$

PRO.

P R O B L E M IV.

To find the value of the reversion of the longest of any number of lives, A, B, C, after the longest of any number of other lives, P, Q, R.

S O L U T I O N.

From the value of all the lives, A, B, C, P, Q, R, subtract the value of the lives, P, Q, R, in possession; the remainder will be the value of the reversion.

The truth of this Solution is almost self-evident; for the excess of the value of all the lives, above that of the lives in possession, is equal to the sum that ought to be paid, for the chance of enjoying the annuity after the decease of these last lives; and so must be the true value of the reversion.

P R O B L E M V.

To find the value of the reversion of any number of joint lives, A, B, C, after any number of joint lives, P, Q, R.

SOLUTION.

From the value of the joint lives in reversion, subtract the value of all the joint lives, and there will remain the value of the reversion.

DEMONSTRATION.

Let the right of the reversion, or all the rents that may happen to arise from the annuity, during the joint continuance of A, B, C, after one of the lives P, Q, R, in possession, is extinct, belong entirely to one person K, and his heirs; and let him be admitted into immediate possession of the annuity, for the joint lives, A, B, C, upon condition that he or his heirs shall pay back the rent thereof, till such time as it becomes his or their own proper right. This being premised, it will appear, that as long as all the lives, A, B, C, P, Q, R, are in being, and no longer, ought K, or his heirs, to pay back the rents of the annuity; for first he ought to pay, while A, B, C, P, Q, R, are all in being, because all this time he receives the rent of an annuity, to which he has no right; but secondly, he ought not to pay after the decease of any of the
lives,

lives, P, Q, R, since whatever he may happen to receive afterwards, is his own just property; nor ought he to pay after the decease of any of the lives, A, B, C, because then he is wholly exempted from all further benefit arising from the annuity. Therefore, seeing the value of all the rents that K and his heirs may happen to pay, is the same as the value of all the *joint* lives, A, B, C, P, Q, R, and the value of all that they may receive, or the whole produce of the annuity, the same as that of the *joint* lives, A, B, C, the Solution is manifest.

Otherwise,

Let the probabilities of the life A, continuing 1, 2, 3, &c. years, be denoted by $a, a', a'',$ &c. and those of the life continuing 1, 2, 3, &c. years, by $b, b', b'',$ &c. respectively, &c. In like manner, let the probabilities of the life P, continuing 1, 2, 3, &c. years, be denoted by $p, p', p'',$ &c. and those of the life Q continuing 1, 2, 3, &c. years, by $q, q', q'',$ &c. let the annuity be 1 L, and m be the amount of 1 L in one year, *viz.* principal and interest. Now, the expectation of A, B, C, upon the first year's rent, depends upon these two events; first, that they all continue in being till the end of that year, and secondly, that some
one,

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one, at least, of the other lives, P, Q, R, fails before that time: Wherefore, seeing the probability of the former is abc , and that of the latter $1-pqr$, the probability that both happen, or that A, B, C, shall receive the first year's rent, will be $abc \times 1-pqr$, or $abc-abc pqr$; this, therefore, multiply'd into $\frac{1}{m}$, the present value of 1 L, due at the end of one year, gives $\frac{abc}{m} - \frac{abc pqr}{m}$, for the true value of the expectation of A, B, C, upon the first year's rent; and by the very same way of reasoning, the expectation of A, B, C, on the 2d, 3d, &c. year's rents, will appear to be $\frac{abc}{m^2} - \frac{abc pqr}{m^2}$, $\frac{abc}{m^3} - \frac{abc pqr}{m^3}$, &c. respectively; the sum of all which, or $\frac{abc}{m} + \frac{abc}{m^2} + \frac{abc}{m^3}$, &c. $-\frac{abc pqr}{m} - \frac{abc pqr}{m^2} - \frac{abc pqr}{m^3}$, &c. is, therefore the total value of the reversion; and this, from Problem I, appears to be equal to the value of the joint lives, A, B, C, less the value of all the joint lives, A, B, C, P, Q, R, as was to be proved.

PRO-

PROBLEM VI.

To find the value of the reversion of any number of joint lives, A, B, C, after the longest of any number of other lives, P, Q, R.

SOLUTION.

Let a, b, c , &c. be as in the last Problem: Then, because the probability of all the lives, A, B, C, continuing till the end of the first year, is abc , and that of all the other lives, P, Q, R, failing before that time, $\frac{1-p}{m} \times \frac{1-q}{m} \times \frac{1-r}{m}$, the probability that both these events happen, or that A, B, C, receive the first year's rent, will (by the preceding Lemma) be $abc \times \frac{1-p}{m} \times \frac{1-q}{m} \times \frac{1-r}{m}$; this therefore multiplied by $\frac{1}{m}$, the present value of 1 L certain, to be received at the end of one year, gives $\frac{abc \times \frac{1-p}{m} \times \frac{1-q}{m} \times \frac{1-r}{m}}{m}$ for the true value of the expectation of A, B, C, upon the first year's rent, allowing for discount, and all contingencies. After the very same manner it will appear, that the value of the expectation of A, B, C, on the 2d, 3d, &c. year's

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year's rents, will be $\frac{a' b' c' \times 1 - p' \times 1 - q' \times 1 - r'}{m'}$, and $\frac{a'' b'' c'' \times 1 - p'' \times 1 - q'' \times 1 - r''}{m'}$, &c. respectively. Therefore the sum of these values, or $\frac{a' b' c' \times 1 - p' \times 1 - q' \times 1 - r'}{m'} + \frac{a'' b'' c'' \times 1 - p'' \times 1 - q'' \times 1 - r''}{m'}$ + $\frac{a''' b''' c''' \times 1 - p''' \times 1 - q''' \times 1 - r'''}{m'}$, &c. is the whole value of the reversion. Q. E. I.

COROLLARY.

If the last general expression be reduced to simple terms, and compared with that in Problem I. it will appear that the value of the reversion, of any number of *joint* lives A, B, C, after the longest of any number of lives, P, Q, R, is equal to the value of all the *joint* lives A, B, C, *less* the values of all the *joint* lives arising from combining (at each combination) all the lives A, B, C, with each *one* of the other; *more* the values of all the *joint* lives, arising from combining all the lives A, B, C, with each *two* of the other; *less* the values of all the *joint* lives, arising from combining all the lives A, B, C, with each *three* of the other, and *so* on.

P R O-

P R O B L E M VII.

To find the value of the reversion of the longest of any number of lives A, B, C, after any number of joint lives, P, Q, R.

S O L U T I O N.

Let $m, a, b, p, q, \&c.$ be still as in the preceding propositions. Therefore, seeing the probability that one or more of the lives P, Q, R, fails the first year, is express'd by $1-pqr$, and that of one of the lives A, B, C, at least, surviving the first year, by $1-1-a \times 1-b \times 1-c$, the probability that these last A, B, C, receive the first year's rent, will be

$1-pqr \times 1-1-a \times 1-b \times 1-c$, and consequently the value of their expectation on

that year's rent, $1-pqr \times \frac{1-1-a \times 1-b \times 1-c}{m}$,

or $\frac{1-1-a \times 1-b \times 1-c}{m} \times \frac{pqr \times 1-1-a \times 1-b \times 1-c}{m}$.

And, from the very same way of reasoning, the value of their expectation on the 2d, 3d, &c. year's rents, will appear to be

$\frac{1-1-a \times 1-b \times 1-c}{m^2} \times \frac{pqr \times 1-1-a \times 1-b \times 1-c}{m^2}$, and

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$\frac{1 - \frac{a}{m} \times 1 - \frac{b}{m} \times 1 - c}{m^3} - \frac{pqr \times 1 - \frac{a}{m} \times 1 - \frac{b}{m} \times 1 - c}{m^4},$
 &c. Therefore the sum all these, or
 $\frac{1 - \frac{a}{m} \times 1 - \frac{b}{m} \times 1 - c}{m} + \frac{1 - \frac{a}{m} \times 1 - \frac{b}{m} \times 1 - c}{m^2} +$
 $\frac{1 - \frac{a}{m} \times 1 - \frac{b}{m} \times 1 - c}{m^3}, \&c. - \frac{pqr \times 1 - \frac{a}{m} \times 1 - \frac{b}{m} \times 1 - c}{m}$
 $- \frac{pqr \times 1 - \frac{a}{m} \times 1 - \frac{b}{m} \times 1 - c}{m^2} -$
 $\frac{pqr \times 1 - \frac{a}{m} \times 1 - \frac{b}{m} \times 1 - c}{m^3}, \&c. \text{ must be the}$
 total value of the reversion. Q. E. I.

COROLLARY.

Hence, because it appears from the preceding problems, that the first series,
 $\frac{1 - \frac{a}{m} \times 1 - \frac{b}{m} \times 1 - c}{m} + \frac{1 - \frac{a}{m} \times 1 - \frac{b}{m} \times 1 - c}{m^2}, \&c.$
 expresses the value of the longest of the lives A, B, C, and that the second
 $\frac{pqr \times 1 - \frac{a}{m} \times 1 - \frac{b}{m} \times 1 - c}{m}, \&c. \text{ (reduced to}$
 simple terms) is equal to the values of all the joint lives arising from combining, at each combination, all the lives P, Q, R, with each one of the other, &c. it follows, that the value of the reversion of any number
of

of lives A, B, C, after any number of *joint* lives, P, Q, R, is equal to the value of the lives A, B, C; *less* the values of all the *joint* lives, arising from the combination of all the lives, in possession with each *one* of the other; *more* the values of all the *joint* lives, arising from the combination of all the lives, in possession with each *two* of the other; *less* the values of all the *joint* lives, arising from the combination of all the lives, in possession with each *three* of the other, and so on.

T A B L E S

For the VALUATION of

ANNUITIES,

Upon One, Two, or Three
L I V E S;

Deduced from ten Years OBSERVATIONS
on the BILLS of MORTALITY of the
City of LONDON.

TABLE I.

For the Valuation of Annuities upon one
LIFE.

Age.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.	Age.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.
6	14.1	16.2	18.8	21	12.9	14.7	17.0
7	14.2	16.3	18.9	22	12.7	14.5	16.8
8	14.3	16.4	19.0	23	12.6	14.3	16.5
9	14.3	16.4	19.0	24	12.4	14.1	16.3
10	14.3	16.4	19.0	25	12.3	14.0	16.1
11	14.3	16.4	19.0	26	12.1	13.8	15.9
12	14.2	16.3	18.9	27	12.0	13.6	15.6
13	14.1	16.2	18.7	28	11.8	13.4	15.4
14	14.0	16.0	18.5	29	11.7	13.2	15.2
15	13.9	15.8	18.3	30	11.6	13.1	15.0
16	13.7	15.6	18.1	31	11.4	12.9	14.8
17	13.5	15.4	17.9	32	11.3	12.7	14.6
18	13.4	15.2	17.6	33	11.2	12.6	14.4
19	13.2	15.0	17.4	34	11.0	12.4	14.2
20	13.0	14.8	17.2	35	10.9	12.3	14.1

Age.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.	Age.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.
36	10.8	12.1	13.9	56	8.4	9.1	10.1
37	10.6	11.9	13.7	57	8.2	8.9	9.9
38	10.5	11.8	13.5	58	8.1	8.7	9.6
39	10.4	11.6	13.3	59	8.0	8.6	9.4
40	10.3	11.5	13.2	60	7.9	8.4	9.2
41	10.2	11.4	13.0	61	7.7	8.2	8.9
42	10.1	11.2	12.8	62	7.6	8.1	8.7
43	10.0	11.1	12.6	63	7.4	7.9	8.5
44	9.9	11.0	12.5	64	7.3	7.7	8.3
45	9.8	10.8	12.3	65	7.1	7.5	8.0
46	9.7	10.7	12.1	66	6.9	7.3	7.8
47	9.5	10.5	11.9	67	6.7	7.1	7.6
48	9.4	10.4	11.8	68	6.6	6.9	7.4
49	9.3	10.2	11.6	69	6.4	6.7	7.1
50	9.2	10.1	11.4	70	6.2	6.5	6.9
51	9.0	9.9	11.2	71	6.0	6.3	6.7
52	8.9	9.8	11.0	72	5.8	6.1	6.5
53	8.8	9.6	10.7	73	5.6	5.9	6.2
54	8.6	9.4	10.5	74	5.4	5.6	5.9
55	8.5	9.3	10.3	75	5.2	5.4	5.6

TABLE II.

*For the the Valuation of Annuities upon
two joint LIVES.*

Mean Age.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.	Mean Age.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.
6	11.3	12.7	14.4	21	10.0	11.2	12.6
7	11.5	12.9	14.6	22	9.8	11.0	12.4
8	11.6	13.0	14.7	23	9.7	10.8	12.2
9	11.6	13.0	14.7	24	9.5	10.6	12.0
10	11.6	13.0	14.7	25	9.4	10.5	11.8
11	11.5	12.9	14.6	26	9.2	10.3	11.6
12	11.4	12.8	14.5	27	9.1	10.1	11.4
13	11.3	12.7	14.3	28	8.9	9.9	11.2
14	11.2	12.5	14.1	29	8.8	9.8	11.0
15	11.0	12.3	13.9	30	8.6	9.6	10.8
16	10.8	12.1	13.7	31	8.5	9.4	10.6
17	10.7	11.9	13.5	32	8.3	9.2	10.4
18	10.5	11.7	13.2	33	8.2	9.1	10.2
19	10.3	11.5	13.0	34	8.1	8.9	10.0
20	10.1	11.3	12.8	35	8.0	8.8	9.9

Mean Age.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.	Mean Age.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.
36	7.8	8.6	9.7	56	5.6	6.1	6.7
37	7.6	8.4	9.5	57	5.5	6.0	6.6
38	7.5	8.3	9.3	58	5.4	5.8	6.4
39	7.4	8.2	9.2	59	5.3	5.7	6.3
40	7.3	8.1	9.1	60	5.2	5.6	6.1
41	7.2	8.0	8.9	61	5.1	5.5	6.0
42	7.1	7.8	8.7	62	5.0	5.4	5.9
43	7.0	7.7	8.6	63	4.9	5.3	5.7
44	6.9	7.6	8.5	64	4.8	5.1	5.5
45	6.7	7.4	8.3	65	4.7	5.0	5.4
46	6.6	7.3	8.2	66	4.6	4.9	5.3
47	6.5	7.2	8.1	67	4.5	4.8	5.1
48	6.4	7.1	7.9	68	4.4	4.6	4.9
49	6.3	7.0	7.8	69	4.3	4.5	4.8
50	6.2	6.8	7.6	70	4.2	4.4	4.6
51	6.1	6.7	7.4	71	4.1	4.3	4.5
52	6.0	6.6	7.3	72	3.9	4.1	4.3
53	5.9	6.5	7.2	73	3.8	4.0	4.2
54	5.8	6.3	7.0	74	3.7	3.8	4.0
55	5.7	6.2	6.9	75	3.6	3.7	3.8

TABLE III.

*For the Valuation of Annuities upon the
longest of two LIVES.*

Mean Age.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.	Mean Age.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.
6	16.9	19.7	23.3	21	15.6	18.2	21.3
7	17.0	19.8	23.4	22	15.4	18.0	21.1
8	17.1	19.9	23.5	23	15.3	17.8	20.8
9	17.1	19.9	23.5	24	15.1	17.6	20.6
10	17.1	19.9	23.5	25	15.0	17.4	20.3
11	17.1	19.9	23.5	26	14.9	17.3	20.1
12	17.0	19.8	23.4	27	14.7	17.1	19.9
13	16.9	19.7	23.3	28	14.6	16.9	19.7
14	16.7	19.5	23.1	29	14.5	16.8	19.5
15	16.6	19.3	22.9	30	14.4	16.6	19.3
16	16.4	19.1	22.6	31	14.2	16.4	19.1
17	16.2	18.9	22.4	32	14.1	16.2	18.9
18	16.1	18.7	22.1	33	14.0	16.1	18.7
19	15.9	18.5	21.9	34	13.9	15.9	18.5
20	15.7	18.3	21.6	35	13.8	15.8	18.3

Mean Age.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.	Mean Age.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.
36	13.7	15.6	18.1	56	11.2	12.1	13.4
37	13.6	15.5	17.9	57	11.0	11.9	13.1
38	13.5	15.3	17.7	58	10.9	11.7	12.8
39	13.4	15.2	17.5	59	10.7	11.5	12.5
40	13.3	15.0	17.3	60	10.5	11.2	12.2
41	13.2	14.9	17.0	61	10.3	11.0	12.0
42	13.1	14.7	16.8	62	10.1	10.8	11.7
43	13.0	14.5	16.5	63	9.9	10.5	11.4
44	12.9	14.3	16.3	64	9.7	10.3	11.1
45	12.8	14.2	16.1	65	9.4	10.0	10.8
46	12.6	14.0	15.8	66	9.2	9.7	10.5
47	12.5	13.8	15.6	67	8.9	9.4	10.2
48	12.4	13.6	15.3	68	8.7	9.2	9.9
49	12.2	13.4	15.1	69	8.5	8.9	9.5
50	12.1	13.3	14.9	70	8.2	8.6	9.2
51	11.9	13.1	14.6	71	8.0	8.4	8.9
52	11.8	12.9	14.4	72	7.7	8.1	8.6
53	11.6	12.7	14.1	73	7.5	7.8	8.2
54	11.5	12.5	13.9	74	7.2	7.5	7.9
55	11.3	12.3	13.6	75	6.9	7.2	7.6

TABLE III.

*For the Valuation of Annuities upon the
longest of two LIVES.*

Mean Age.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.	Mean Age.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.
6	16.9	19.7	23.3	21	15.6	18.2	21.3
7	17.0	19.8	23.4	22	15.4	18.0	21.1
8	17.1	19.9	23.5	23	15.3	17.8	20.8
9	17.1	19.9	23.5	24	15.1	17.6	20.6
10	17.1	19.9	23.5	25	15.0	17.4	20.3
11	17.1	19.9	23.5	26	14.9	17.3	20.1
12	17.0	19.8	23.4	27	14.7	17.1	19.9
13	16.9	19.7	23.3	28	14.6	16.9	19.7
14	16.7	19.5	23.1	29	14.5	16.8	19.5
15	16.6	19.3	22.9	30	14.4	16.6	19.3
16	16.4	19.1	22.6	31	14.2	16.4	19.1
17	16.2	18.9	22.4	32	14.1	16.2	18.9
18	16.1	18.7	22.1	33	14.0	16.1	18.7
19	15.9	18.5	21.9	34	13.9	15.9	18.5
20	15.7	18.3	21.6	35	13.8	15.8	18.3

Mean Age.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.	Mean Age.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.
36	13.7	15.6	18.1	56	11.2	12.1	13.4
37	13.6	15.5	17.9	57	11.0	11.9	13.1
38	13.5	15.3	17.7	58	10.9	11.7	12.8
39	13.4	15.2	17.5	59	10.7	11.5	12.5
40	13.3	15.0	17.3	60	10.5	11.2	12.2
41	13.2	14.9	17.0	61	10.3	11.0	12.0
42	13.1	14.7	16.8	62	10.1	10.8	11.7
43	13.0	14.5	16.5	63	9.9	10.5	11.4
44	12.9	14.3	16.3	64	9.7	10.3	11.1
45	12.8	14.2	16.1	65	9.4	10.0	10.8
46	12.6	14.0	15.8	66	9.2	9.7	10.5
47	12.5	13.8	15.6	67	8.9	9.4	10.2
48	12.4	13.6	15.3	68	8.7	9.2	9.9
49	12.2	13.4	15.1	69	8.5	8.9	9.5
50	12.1	13.3	14.9	70	8.2	8.6	9.2
51	11.9	13.1	14.6	71	8.0	8.4	8.9
52	11.8	12.9	14.4	72	7.7	8.1	8.6
53	11.6	12.7	14.1	73	7.5	7.8	8.2
54	11.5	12.5	13.9	74	7.2	7.5	7.9
55	11.3	12.3	13.6	75	6.9	7.2	7.6

TABLE IV.

For the Valuation of Annuities upon three joint LIVES.

Mean Age.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.	Mean Age.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.
6	9.7	10.6	11.7	21	8.2	9.0	10.0
7	9.9	10.8	11.9	22	8.1	8.9	9.8
8	10.0	10.9	12.0	23	7.9	8.7	9.6
9	10.0	10.9	12.0	24	7.7	8.5	9.4
10	10.0	10.9	12.0	25	7.6	8.3	9.2
11	9.9	10.8	11.9	26	7.4	8.1	9.0
12	9.8	10.7	11.8	27	7.3	8.0	8.8
13	9.6	10.5	11.6	28	7.1	7.8	8.6
14	9.5	10.4	11.4	29	7.0	7.7	8.5
15	9.3	10.2	11.2	30	6.8	7.5	8.3
16	9.2	10.0	11.0	31	6.7	7.4	8.2
17	9.0	9.8	10.8	32	6.5	7.2	8.0
18	8.8	9.6	10.6	33	6.4	7.1	7.9
19	8.6	9.4	10.4	34	6.2	6.9	7.7
20	8.4	9.2	10.2	35	6.1	6.8	7.6

Mean Age.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.	Mean Age.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.
36	6.0	6.7	7.4	56	4.4	4.7	5.1
37	5.9	6.5	7.2	57	4.3	4.6	5.0
38	5.8	6.4	7.1	58	4.2	4.5	4.9
39	5.7	6.3	7.0	59	4.1	4.4	4.8
40	5.6	6.2	6.9	60	4.0	4.3	4.6
41	5.5	6.1	6.8	61	3.9	4.2	4.5
42	5.4	6.0	6.7	62	3.8	4.1	4.4
43	5.4	5.9	6.5	63	3.7	4.0	4.3
44	5.3	5.8	6.4	64	3.7	3.9	4.2
45	5.2	5.7	6.3	65	3.6	3.8	4.1
46	5.1	5.6	6.2	66	3.5	3.7	3.9
47	5.0	5.5	6.1	67	3.4	3.6	3.8
48	5.0	5.4	5.9	68	3.3	3.5	3.7
49	4.9	5.3	5.8	69	3.2	3.4	3.6
50	4.8	5.2	5.7	70	3.1	3.2	3.4
51	4.7	5.1	5.6	71	3.0	3.1	3.3
52	4.7	5.1	5.5	72	2.9	3.0	3.1
53	4.6	5.0	5.4	73	2.8	2.9	3.0
54	4.5	4.9	5.3	74	2.6	2.7	2.8
55	4.4	4.8	5.2	75	2.5	2.6	2.7

TABLE V.

*For the Valuation of Annuities upon the
longest of three LIVES.*

MeanAge.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.	MeanAge.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.
6	18.0	21.0	25.0	21	16.9	19.5	23.1
7	18.1	21.1	25.1	22	16.8	19.4	22.8
8	18.2	21.2	25.2	23	16.6	19.2	22.6
9	18.2	21.2	25.2	24	16.5	19.0	22.3
10	18.2	21.2	25.2	25	16.4	18.8	22.1
11	18.2	21.2	25.2	26	16.3	18.7	21.9
12	18.1	21.1	25.1	27	16.1	18.5	21.6
13	18.0	21.0	25.0	28	16.0	18.3	21.4
14	17.9	20.9	24.8	29	15.9	18.2	21.2
15	17.8	20.7	24.6	30	15.8	18.0	21.0
16	17.6	20.5	24.3	31	15.6	17.8	20.8
17	17.5	20.3	24.1	32	15.5	17.7	20.6
18	17.3	20.1	23.8	33	15.4	17.6	20.4
19	17.2	19.9	23.5	34	15.3	17.4	20.2
20	17.0	19.7	23.3	35	15.2	17.3	20.0

Mean Age.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.	Mean Age.	Year's Purch. at 5 per cent.	Year's Purch. at 4 per cent.	Year's Purch. at 3 per cent.
36	15.1	17.2	19.9	56	12.6	13.7	15.1
37	15.0	17.0	19.7	57	12.5	13.5	14.8
38	14.9	16.9	19.5	58	12.3	13.2	14.5
39	14.8	16.7	19.3	59	12.1	12.9	14.1
40	14.7	16.6	19.1	60	11.9	12.7	13.8
41	14.6	16.4	18.9	61	11.7	12.5	13.5
42	14.5	16.3	18.7	62	11.5	12.2	13.1
43	14.4	16.2	18.5	63	11.3	11.9	12.8
44	14.3	16.0	18.2	64	11.0	11.6	12.5
45	14.2	15.9	18.0	65	10.8	11.4	12.2
46	14.1	15.7	17.7	66	10.5	11.1	11.8
47	13.9	15.5	17.5	67	10.2	10.8	11.5
48	13.8	15.3	17.2	68	9.9	10.5	11.2
49	13.7	15.1	17.0	69	9.6	10.2	10.9
50	13.5	14.9	16.7	70	9.3	9.9	10.5
51	13.4	14.7	16.5	71	9.0	9.6	10.2
52	13.2	14.5	16.2	72	8.7	9.2	9.8
53	13.1	14.3	15.9	73	8.4	8.9	9.5
54	12.9	14.1	15.7	74	8.1	8.6	9.1
55	12.8	13.9	15.4	75	7.8	8.2	8.7

48 *Of the VALUATION of*

Here follow the practical Solutions of several Problems, depending on the foregoing tables.

PROBLEM VIII.

To find the value of an annuity for an assigned life.

SOLUTION.

Look out the given age in table I. and against it, towards the right-hand, under the proposed rate of interest, will stand the number of years purchase, which an annuity upon that life is worth.

EXAMPLE.

Let the given age be 18 years, and the rate of interest 4 *per cent.* then looking against 18, under 4 *per cent.* I find 15.2, equal the number of years purchase required.

PROBLEM IX.

To find the value of an annuity upon two assigned joint lives.

SO-

SOLUTION.

C A S E I.

If the two lives be equal; enter tab. II. with the common age, and against it you will have the value required.

C A S E II.

If the given ages be unequal, but neither of them less than 25, nor greater than 50 years; take half the sum of the two for a mean age, and proceed as in Case I *.

C A S E III.

If one or both ages be without the limits abovemention'd, but so that the difference of the values corresponding to those ages, be not more than $\frac{1}{3}$ of the lesser; let $\frac{4}{10}$ of that difference be added to the said lesser value, and the sum will be the value sought.

* This and the following Solutions are so contriv'd, as to be always depended on to less than $\frac{1}{4}$ of a year's purchase, as I shall hereafter endeavour to make appear.

50 *Of the VALUATION of*

Generally,

Be the difference of the values what it will, multiply it by $\frac{1}{2}$ the lesser of the two values, dividing the product by the greater; then the quotient, added to the lesser value, will give the true answer very near.

EXAMPLE of CASE I.

Let the two given ages be each 18, and interest at 5 *per cent.* then in tab. II. against 18, under 5 *per cent.*, is 10.5 years purchase.

EXAMPLE of CASE II.

In which the rate of interest is supposed as above, and one of the two ages 34, the other 48; therefore the half sum of the ages is 42, against which stands 7.1.

EXAMPLE of CASE III.

Where one age is supposed to be 15 years, the other 29; here against 15 years will be found 11.0, and against 29, 8.8, the difference of which two values is 2.2,
and

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and $\frac{4}{10}$ thereof, equal to 0.88; this therefore, added to 8.8, gives 9.68, or 9.7, for the answer.

EXAMPLE of CASE IV.

Let the rate of interest be 4 per cent. and one age 11 years, the other 68. The values corresponding to these ages, are 12.9. and 4.6, their difference is 8.3, which multiply'd by 2.3. will be 19.09, this divided by 12.9, quotes 1.5, which therefore, added to 4.6, the lesser value, gives 6.1, equal the value sought.

PROBLEM X.

To find the value of an annuity upon two lives, that is, to continue as long as either of them is in being.

SOLUTION,

CASE I,

If the lives be equal, find the given age in tab. III. and against it, under the proposed rate of interest, will be the number of years purchase required.

C A S E II.

If both ages be between 25 and 50, take half their sum for a *mean age*, and proceed as in Case I.

C A S E III.

If one or both ages be without the limits mentioned in the last case, but the difference of values corresponding to those ages, as found in tab. III. be not more than $\frac{1}{6}$ part of the lesser; take half the sum of those values for the value required.

Generally,

Let the given ages be what they will, find the value of the two joint lives by Case IV. Prob. IX. which subtract from the sum of the values of ~~the~~ two single lives, and there will remain the required value of an annuity upon the longest life.

EXAMPLE of CASE I,

Wherein the two given ages are each supposed 50 years, and the rate of interest 4 *per cent.* Here against 50 years, in tab.

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tab. III. under 4 *per cent.* stands 13.3, shewing the number of years purchase, which an annuity is worth for two such lives.

EXAMPLE of CASE II.

Suppose one age 30 years, and the other 46; then, the half sum of the ages will be 38, answering to which, under 4 *per cent.* stands 15.3.

EXAMPLE of CASE III.

Let the two proposed ages be 6, and 21 years; then against 8 years will be 19.7, and against 21, 18.2, the half sum whereof, is 18.95, equal to the number of years purchase required.

EXAMPLE of CASE IV.

Let one age be 11 years, the other 68, and the rate of interest as in the preceding examples. Then the value of the two *joint* lives, by Case IV. of the last Problem, will be found 6.1, and the values of the single lives, by Problem VIII. equal to 16.3, and 6.7, the sum of which two, decreased by 6.1, is 16.9, equal to the value required.

PROBLEM XI.

To find the value of an annuity upon three joint lives.

SOLUTION.

CASE I.

If all the lives be equal; find out the given age in tab. IV. and against it, under the proposed rate of interest, will be the number of years purchase required.

CASE II.

If all the three ages be between 15 and 55 years, and the difference between the greatest and least of them not more than 15 years, take $\frac{1}{3}$ part of their sum for the *mean age*, and proceed as in Case I.

CASE III.

If one or more of the proposed ages be without the limits, mentioned in the last article, but the difference of the values answering to the greatest and least of

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of them, not greater than half the least; then to the sum of the two greater values, add twice the least, and take $\frac{1}{4}$ of the sum for a mean value required.

Generally,

Be the ages what they will, multiply the sum of the three corresponding values, by the square of the least of them, reserving the product; multiply the two greater values into each other, and to the double of the product, add the square of the lesser value; divide the reserved product by this sum, and subtract the quotient from twice the lesser value; the result will be the value sought.

EXAMPLE of CASE I.

Let each age be 35, and the rate of interest 3 *per cent.* then in tab IV. against 35, under 3 *per cent.* stands 7.6, which is the number of years purchase that an annuity is worth for the three *joint* lives.

EXAMPLE of CASE II.

Let the three given ages be 20, 25, and 33 years. Here $\frac{1}{3}$ of the ages will be 26, corref-

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 corresponding to which, under 3 *per cent.*
 stands 9.0.

EXAMPLE of CASE III.

Where the proposed ages are 7, 15;
 and 33 years; against these stand 11.9,
 11.2, and 7.9, therefore the sum of the
 two greater values is, here, 23.1; this added
 to twice the lesser, gives 38.9, the $\frac{1}{4}$ of
 which, or 9.725, is the value sought.

EXAMPLE of CASE IV.

Let the three ages be 13, 31 $\frac{1}{2}$, and 53
 years, and interest 4 *per cent.* then the va-
 lues answering to those ages, will be 10.5,
 7.3, and 5.0; the sum whereof is 22.8,
 which multiply'd by 25, the square of the
 least of them gives 570, to be reserved;
 Again, the two greatest values multiply'd
 into each other produce, 76.65, the dou-
 ble of this added to 25, the square of the
 least will be 178.3, by which dividing 570,
 the reserved product, there comes out 3.2;
 this subtracted from 10, the double of the
 least value, leaves 6.8 for the value re-
 quired.

PRO-

PROBLEM XII.

To find the value of an annuity upon the longest of three lives.

SOLUTION:

CASE I.

If the lives be all equal, seek the common age in tab. V. and against it, under the proposed rate of interest, will be the number of years purchase required.

CASE II.

If none of the ages be less than 10, nor greater than 60 years, and the difference between the greatest and least of them not more than 15 years, to twice the sum of the two least add the greatest, and take $\frac{1}{5}$ part of the sum as a *mean age*.

CASE III.

If the difference of the greatest and least values, found against the proposed ages in tab. V. be not more than $\frac{1}{4}$ of the least; then, to twice the sum of the

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two

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two greatest values, add the least; taking $\frac{1}{5}$ part of the sum for a *mean value*.

Generally,

Find the value answering to the greatest of the given ages in tab. III. and the values corresponding to all the three several ages in tab. V. and let the difference of the two values, answering to the greatest age, be taken and reserved; let the square of the greater of these two, be divided by the product of the two other remaining values; multiply the square of the quotient by the reserved difference, then this last product, added to the value of an annuity for the two youngest lives, will be the value required.

EXAMPLE of CASE I.

Let the three ages be each 35 years, and interest 4 *per cent.* then in tab. V. against 35, under 4 *per cent.* stands 17.3, for the number of years purchase required.

EXAMPLE of CASE II.

Let the proposed ages be 16, 24, and 30 years, then will the *mean age* be 22 years, and the number of years purchase required 19.4.

Ex.

EXAMPLE of CASE III.

Suppose the three ages to be 28, 35, and 44, then the three corresponding values will be 18.3, 17.3, and 16.0, and therefore twice the sum of the two greater added to the lesser, is 87.2, which divided by 5, quotes 17.44 for the answer.

EXAMPLE of CASE IV.

Let the given ages be 20, 36, and 60, and interest as in the preceding examples: Here, the value found against 60 years in tab. III. is 11.2, and those against 20, 36, and 60, in tab. V. 19.7, 17.2, and 12.7, respectively; wherefore, taking 11.2 from 12.7, we have 1.5 for the difference to be reserved: Now the square of 12.7, divided by the product of 19.7, and 17.2 is 0.5, the square of which, multiply'd by 1.5, the reserved difference gives 0.375; this added to 17.0, the value of an annuity for the two youngest lives (as determined by Case II. Prob. X.) will give 17.375, or 17.4 for the number of years purchase, which an annuity is worth upon all the three lives.

R E M A R K.

That the reader may not entertain any scruple concerning the exactness of the methods of Solution hitherto laid down, for estimating the values of annuities upon two or more unequal lives, I shall here, according to my promise, endeavour to make it appear, that those Solutions may be always depended on, as very near the truth. In order to this, it will be requisite to resume the two hypotheses laid down in Corol. II. and IV. Prob. I. wherein the probabilities of life are supposed in a geometrical, and in an arithmetical progression; and to compare the values of *equal* fictitious lives, computed according to those hypotheses, with the corresponding values in the tables, for real lives, computed from actual observations, and then to consider from thence, how the values ought to differ in lives that are *unequal*. Accordingly, let the value of each of the equal lives, whether considered as real or fictitious, be supposed equal to any number of years purchase, as 7, 8, 9, 10, 11, 12, 13, 14, and 15, successively, and let the rate of interest be at 4 *per cent.* then will the corresponding value, of two equal joint lives, be as in the following little table, whereof the first column expresses the value of each of the
single

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single lives, and the 2d, 3d, and 4th columns, the value of the joint lives, according to observations, and the two foresaid hypotheses respectively.

Value of one single life.	Value of 2 joint lives, per tab.	Value of 2 joint lives, per 1st hypoth.	Value of 2 joint lives, per 2d hypoth.
7	4.7	3.9	4.9
8	5.4	4.6	5.6
9	6.1	5.3	6.4
10	6.8	6.1	7.1
11	7.6	6.9	7.9
12	8.5	7.8	8.8
13	9.5	8.7	9.7
14	10.5	9.7	10.6
15	11.5	10.6	11.6

Now, by inspecting this table, we may observe, first, that the value of the joint lives, according to the last of the two hypotheses, is a small matter greater than the value of the same lives, as deduced from real observations, but never by more than about $\frac{3}{10}$ of an year's purchase; and secondly, that, on the other hand, the value of the joint lives, according to the first hypothesis, is always less than the true value deduced from observations, and that

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at least by $\frac{7}{10}$ of a year's purchase. Hence we may infer, that the probabilities of life, as given in the table of observations, do not come so near a geometric progression, as to an arithmetic one (which, in some measure, appears from the table itself) and consequently that the value of an annuity upon real lives, whether equal or unequal, will differ little from the value derived from the last hypothesis, but something more from the former. Let us, therefore, now see what the differences will be, in two unequal joint lives, by the general rule before given, (in Prob. IX.) from whence we shall be enabled to judge of the exactness of that rule. What these differences are, may be seen by the following table, which exhibits the values of the joint lives, according to each of the three fore-said ways; wherein the value computed by the rule, compared with those derived from the hypotheses, appears to agree so exactly, throughout the whole table, with what has been above observed, with respect to the true value, as to sufficiently prove, that the rule itself must be very near the truth. But if this rule be near the truth, the two particular ones preceding it, must be so too, being so contrived, as to always bring out nearly the same value with the general one; but with this difference, that, as the general

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one, for the most part, gives the answer a little too small, the first of these always makes it a little too great; tho' neither of them scarce ever err, by more than $\frac{1}{10}$ of a year's purchase.

Values of the two single lives.	Value of the two joint lives, per rule.	Value of the two joint lives, per first hypoth.	Value of the two joint lives, per second hypoth.
6 and 8	4.5	3.8	4.7
6 10	4.8	4.3	5.0
6 12	5.0	4.6	5.2
6 14	5.2	4.9	5.4
6 16	5.4	5.2	5.5
8 10	6.0	5.3	6.2
8 12	6.4	5.8	6.7
8 14	6.7	6.3	7.0
8 16	6.9	6.7	7.2
10 12	7.5	6.8	7.8
10 14	8.0	7.5	8.3
10 16	8.4	8.1	8.7
12 14	9.3	8.6	9.5
12 16	9.9	9.4	10.1
14 16	11.4	10.6	11.5

In

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In the same manner it may be made to appear, that the other rules for three *joint* lives, and the *longest* of two or three lives, are likewise very near the truth, but I shall content my self here, with giving one or two instances, in annuities upon three joint lives. Let there be three equal lives, and the value of an annuity upon each of them 14 years purchase, and interest at 4 *per cent.* then will the value of the joint lives, by tab. IV. come out 8.3, but by the two hypotheses, 7.3, and 8.5, respectively. Again, let the lives be supposed very unequal, so as to be worth 6, 10, and 16 years purchase, then will the value of the joint lives be, by the general rule, 4.5, but according to the hypotheses, 3.84, and 4.63; which examples, agreeing so well with each other, and with what has been abovesaid, tend greatly to evince the accuracy of the rules, or, at least, to shew that they are consistent with the table of observations.

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REVERSIONS.

PROBLEM XIII.

TO find the value of the Reversion of one life after another.

SOLUTION.

From the value of the life in expectation, subtract the value of the two *joint* lives, or from the value of the *longest* of the two lives, subtract the value of the life in possession; the remainder, in either case, will be the required value of the Reversion *.

* This and all the following Solutions, relating to Reversions, are universally true, be the probability of life what it will, as appears from Prob. IV. V. and VI. they being nothing more than the most useful Cases of the general Theorems there given.

E X A M P L E.

Suppose the age of the life in possession to be 68 years, and that of the life in expectation 11 years, and interest 4 *per cent.* then the values of the two *joint* lives, by Case IV. Prob. IX. will be 6.1, which subtracted from 16.3, the value of the life in expectation, leaves 10.2, for the value of the reversion; but if the youngest life had been in possession, the value of the Reversion would have been only 0.6.

P R O B L E M XIV.

To find the value of the Reversion of two lives after one.

S O L U T I O N.

From the value of the three lives, subtract the value of the life in possession; the remainder will be the value of the two lives in Reversion.

E X A M P L E.

Let the age of the life in possession be 50 years, and those of two lives in reversion,

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45 and 56 years, and interest at 4 *per cent.* then the value of the three lives, by Case II. Prob. XII. will be 15.1, from which subtracting 10.1, the value of the life in possession, there remains 5.0, for the value required.

PROBLEM. XV.

To find the value of the Reversion of one life after two.

SOLUTION.

From the value of the three lives, take the value of the two lives in possession, there will remain the value of the life in reversion.

EXAMPLE.

Suppose 18 and 26 to be the ages of the two lives in possession, and 32 that of the life in expectation, and interest at 4 *per cent.* then the value of the three lives, by Case II. Prob. XII. will be 19.0, from which subtracting 18.0, the value of the two lives in possession, there remains 1.0, for the value of the reversion.

PROBLEM XVI.

To find the value of the reversion of one life after two joint lives.

SOLUTION.

From the value of the life in expectation, subtract the value of the three *joint* lives, there will remain the value of the life in reversion.

EXAMPLE.

Let the age of each of the three proposed lives, be 21 years; then the value of the three *joint* lives, by Case I. Prob. XI. will be 9.0, which subtracted from 14.7, the value of the life in expectation, leaves 5.7, equal the value of the reversion, when interest is at 4 *per cent.*

PROBLEM XVII.

To find the value of the reversion of two joint lives after one.

S O.

SOLUTION.

From the value of the two *joint* lives, subtract the value of the three joint lives; the remainder will be the value of the reversion.

EXAMPLE.

Suppose the age of the life in possession to be 16 years, and the ages of the two joint lives in reversion, each 28 years; then the value of the two *joint* lives will be 9.9, by Case I. Prob. IX. and that of the three *joint* lives, 8.5, by Case II. Prob. XI. the difference of which, or 1.4, is the value of the reversion; interest being 4 *per cent*.

PROBLEM XVIII.

Supposing two persons, A, B, to be equally in possession of an annuity, which after the decease of either of them, is to belong entirely to the survivor for life: To find the value of the share of each in that annuity.

S O-

SOLUTION.

From the value of the life A, or B, subtract half the value of the two *joint* lives; the remainder will be the value of the share of A, or B.

EXAMPLE.

Let the age of A be 18 years, and that of B, 29 years, and the rate of interest 4 *per cent.* then the value of the life A will be 15.2, and the value of the life B 13.2, from each of which subtracting (5.3) half the value of the two *joint* lives, there will remain 9.9, and 7.9, equal to the two values required.

PROBLEM XIX.

A and B enjoy an annuity, to which a third person C, after the decease of A, is to have the sole right of possession for life, provided B be then extinct; otherwise it is to be equally divided between him and B, during their joint lives, and then to belong entirely to C, for life, if he be the last survivor: To find the value of the right of C in that annuity.

S O-

SOLUTION.

From the value of the life C, subtract half the value of the two joint lives B, C, and from the value of the two joint lives A, C, subtract half the value of the three *joint* lives A, B, C, take this last remainder from the former, and the result will be the value sought.

DEMONSTRATION.

Let the annuity be 1 L, r the rate of interest, and a, b, c , &c. a', b', c' , &c. the probabilities of living 1, 2, &c. years, as in the foregoing propositions. Now the expectation of C, upon what he may happen to receive at the end of any year (suppose the first) may be considered in two parts, as depending on two different events; for first, if C be the only person of the three then living, of which the probability is $c \times \overline{1-a} \times \overline{1-b}$, he will be intitled to the whole year's rent, or 1 L, therefore his present expectation thereon, discount being allow'd, is $c \times \overline{1-a} \times \overline{1-b} \times \frac{1}{r}$. Secondly, he and B may happen to be living, and A
only

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only extinct, in which case he is to receive only $\frac{1}{2}$ of 1 L; therefore, the probability of this being $cb \times \overline{1-a}$, his expectation in this case will be $cb \times \overline{1-a} \times \frac{1}{2r}$: Hence, by adding these two values together, we have $c \times \overline{1-a} \times \overline{1-b} \times \frac{1}{r} + cb \times \overline{1-a} \times \frac{1}{2r}$
 $= \frac{c}{r} - \frac{cb}{2r} - \frac{ca}{r} + \frac{abc}{2r}$ for the total expectation of C, upon what he may happen to receive at the end of the first year. And, from the very same manner of reasoning, it will appear that $\frac{c'}{r^2} - \frac{c'b}{2r^2}$
 $- \frac{a'c'}{r^2} + \frac{a'b'c'}{2r^2}$ is the value of the expectation of C, upon what he may happen to receive at the end of the second year, &c. Therefore the total expectation, or present value of all the sums that C may happen to receive from time to time, is

$$\begin{aligned}
 &+ \frac{c}{r} - \frac{cb}{2r} - \frac{ca}{r} + \frac{abc}{2r} \\
 &+ \frac{c'}{r^2} - \frac{c'b}{2r^2} - \frac{c'a}{r^2} + \frac{a'b'c'}{2r^2} \\
 &+ \frac{c''}{r^3} - \frac{c''b}{2r^3} - \frac{c''a}{r^3} + \frac{a''b''c''}{2r^3} \\
 &\text{\&c.}
 \end{aligned}$$

But

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and A, C, and to the remainder add $\frac{1}{3}$ of the value of the three *joint* lives A, B, C, the sum shall be the value which A ought to contribute; and the like will hold with regard to the lives B and C.

DEMONSTRATION.

The expectation of A, upon what he may happen to receive at the end of one year, may be consider'd in four parts, as depending on so many different events; for, first, A, B, and C, may be all living, the probability whereof is abc , in which case he is to receive $\frac{1}{3}$ of the year's rent, or $\frac{1}{3}$ of 1 L; therefore his expectation on this event is $\frac{abc}{3r}$. Secondly, A and B may be living, and C extinct, the probability whereof is $ab \times 1 - c$, therefore on this event his expectation is $\frac{ab \times 1 - c}{2r}$. Thirdly, A and C may be living, and B extinct, on this event his expectation is $\frac{ac \times 1 - b}{2r}$. Lastly, A may be the only person then living, upon this the expectation will be $\frac{a \times 1 - b \times 1 - c}{r}$; wherefore, by adding all

all these four values into one sum, we have $\frac{a}{r} - \frac{ab}{2r} - \frac{ac}{2r} + \frac{abc}{3r}$ for the total value of the expectation of A, upon what he may happen to receive at the end of the first year; from whence, by following the same method of reasoning, as in the last proposition, the truth of the Solution will appear evident.

In like manner may the share of A be determined, be the number of persons concerned in the purchase ever so great, supposing the annuity to be always equally divided among the surviving lives; for let P express the value of the life A (*viz.* the value of an annuity of 1 L for the life of A;) Q the sum of the values of all the *joint* lives, arising from the combination of the life A, with each *one* of the other; R the sum of the values of all the *joint* lives, arising from the combination of the life A, with each *two* of the other; S the sum of the values of all the *joint* lives, arising from the combination of the life A, with each *three* of the other, and so on; then will $P - \frac{Q}{2} + \frac{R}{3} - \frac{S}{4} + \frac{T}{5}$, &c. be the value which A ought to contribute.

P R O B L E M X X I.

Supposing any given number of lives P, Q, R, and that A, or his Heirs, are to receive the sum S upon the first vacancy of any of these lives: To find the value of A's expectation in present money.

S O L U T I O N.

Multiply the given sum by the value of an annuity for the joint lives P, Q, R, and divide the product by the value of the same annuity for ever; subtract the quotient from the given sum, and there will remain the value sought.

D E M O N S T R A T I O N.

Let E be the value of an annuity for ever (*i. e.* the number of years purchase it is worth) and P the value of an annuity for the proposed joint lives; therefore, seeing the value of the reversion for ever, after the joint lives P, Q, R, to be received as soon as one of those lives becomes extinct, is to the sum (S) to be received at the same time, as E to S, the
present

present value of that reversion, must, consequently, be to the present value of this sum, in the same ratio of E to S; but the present value of the reversion is known to be $E - P$, therefore that of the sum S will be $\frac{E-P}{E} \times S = S - \frac{PS}{E}$. Q. E. D.

EXAMPLE.

Let the number of lives be 3, their ages each 27 years, the rate of interest 4 per cent. and the proposed sum 100*l*. then the value of an annuity for the joint lives being (by the table) 8 years purchase, and the value of an annuity for ever 25 years purchase, we shall, by multiplying 100*l*. by 8, and dividing the product by 25, have 32*l*. which subtracted from 100*l*. will leave 68*l*. for the present value of 100*l*. to be received at the first vacancy of the three proposed lives.

PROBLEM XXII.

Supposing a given sum S to be depending, so as not to become due till the lives P; Q, R, &c. are all extinct; 'tis required to find the value of that sum in present money.

S O-

SOLUTION.

Multiply the given sum by the value of an annuity for the longest of the proposed lives, and divide the product by the perpetuity; subtract the quotient from the given sum, and there will remain the value required.

The reasons of this Solution will appear evident from the method laid down in the last Problem.

REMARK.

In what has been hitherto laid down, we have had regard to such annuities as are paid yearly, but if the payments are to be made every half year, which is most commonly the case, the true value at which the purchase is to be estimated, ought to be a little increased; and therefore it may not be improper to consider here, how much that increase will be. In order to which, let E denote the perpetuity, or the value of an annuity for ever, and P the value of the same annuity for any number of lives; then, since that life, upon whose failing the annuity ceases, has the same chance very nearly, to drop in the last as in the first half of the year, it is manifest that

that the purchaser has, in this case, an equal chance to receive half a year's rent more than when the annuity is paid yearly; and therefore the sum that ought to be allow'd as an equivalent, for this, to be paid upon ceasing of the annuity, will be $\frac{1}{4}$ of a year's purchase, or $\frac{1}{4} L$, of which sum, the present value will, by the last Prob. be $\frac{E-P}{4E}$; but this is not the only advantage the purchaser has in this case; for, besides this, he has the use of the 1st, 3d, 5th, &c. half years rents each half a year; the value of which consideration, since the interest of $\frac{1}{2} L$, for half a year, is to 1 L, the whole sum received in that year, in the constant proportion of 1 to 4 E, will, it is manifest, be to the present value (P) of all the sums that may be received from time to time, in the same constant proportion; and therefore is equal to $\frac{P}{4E}$, which added to $\frac{E-P}{4E}$, above found, gives $\frac{1}{4} L$ for the whole value required. Hence it appears, that be the rate of interest and the number of lives what they will, the difference of the values of two equal annuities, arising from the rents of one being paid yearly, and the other half yearly,

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yearly, will in all cases be $\frac{1}{4}$ of a year's purchase; and in like manner, it will be found, that the difference of the values of two annuities, upon account of the rents of the one being paid yearly, and the other quarterly, will, in all cases be $\frac{3}{8}$ of a year's purchase very nearly.

From the same method of proceeding, may the difference between the value of an annuity payable in money, and the value of an estate in land, where the purchaser enjoys it, by an actual possession, to the last moments of life, be determined; for, since it is an equal chance whether the life on which the estate last depends, drops in the former or latter half of the year, the purchaser may be supposed, in this case, to enjoy an annuity half a year longer, than he who is paid in money yearly, and intirely loses the last payment, if he dies but one day before it becomes due; and therefore the present value of this consideration, by the last Prob. will be $\frac{E-P}{E} \times \frac{1}{2}$; whence it appears, that if from the perpetuity you subtract the value of an annuity payable in money, and divide the remainder by double the perpetuity, the result will be the parts of a year's purchase, expressing the difference of values required.

This

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This is true, when the value of the estate is to be estimated from the yearly rent it would produce, or when the possession for one whole year is exactly equivalent to 1 L, to be received at the end of that year; but if the estate, when let, will bring in $\frac{1}{2}$ L every half year, then there being the advantage of the interest of the 1st, 3d, 5th, &c. payments, for half a year each, the value $\frac{E-P}{2E}$, above found, ought to be increased by $\frac{P}{4E}$, which will bring it to $\frac{1}{2} - \frac{P}{4E}$, for the difference required here, which in most real cases will be $= \frac{3}{8}$ of a year's purchase very near. Hence we may conclude, that if to the value of the annuity, as found in the tables, be added $\frac{1}{4}$ of a year's purchase, when the rents are to be paid half-yearly, or $\frac{3}{8}$ of a year's purchase when they are to be paid quarterly, or the purchase is in land; the sum will give the value of the annuity in those cases respectively.

M

P R O.

P R O B L E M X X I I I .

There is an estate, which, if A, the present possessor, happens to die in a given time, or before he attains to a certain age, is afterwards to belong to B, and his heirs for ever; to find the value of B's expectation.

S O L U T I O N .

The expectation of B may be considered in two parts, one with regard to what he may receive during the proposed term, and the other with regard to what he may receive after that term is expired; the former of which, as all the rents that become due in that term, will belong either to him or A, must, it is manifest, be equal to the difference between the value of an annuity *certain* for (n) the number of years in the said term, and the value of another equal annuity for the same number of years, as depending on the contingency of A's continuing alive to the end of that time; but the latter part of B's expectation, since the whole estate is to belong to him and his heirs for ever, if A should happen to be extinct at the end of the aforesaid term,

term,

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term, will be the value of the perpetuity to be received n years hence, multiply'd by the probability of A's not living to the end of that time; wherefore, if this probability be denoted by P , the perpetuity by E , and the present value of 1 L, due n years hence, by m , and $r, a, a', a'', \&c.$ be supposed as in the preceding propositions, it is evident from what has been abovesaid, that $p m E$

$$+ \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4} + \frac{1}{r^5} + \frac{1}{r^6}, \&c.$$

$$- \frac{a}{r} - \frac{a'}{r^2} - \frac{a''}{r^3} - \frac{a'''}{r^4} - \frac{a^{(4)}}{r^5} - \frac{a^{(5)}}{r^6}, \&c.$$

or,

$$p m E + \frac{1-r^{-n}}{r-1}$$

$$- \frac{a}{r} - \frac{a'}{r^2} - \frac{a''}{r^3} - \frac{a'''}{r^4}, \&c.$$

will be the total value of B's expectation; where each series is to be continued to as many terms, as there are units in n . But as the finding and adding together those terms, when their number is great, may seem to require too much labour for common practice, I have deduced the following rule therefrom; which, being applicable to the tables, gives the true answer without any great trouble.

R U L E.

From the perpetuity subtract the value of a life of that age, to which the expectation of B is limited, and multiply the remainder by the probability that A lives to that age, and this product again, by the present value of 1 L, to be received at the end of the given term; to this last product add the value of the life A, in possession, and take the sum from the perpetuity, and there will remain the value required.

E X A M P L E.

Let the age of A be 8 years, and interest at 5 *per cent.* and let B be intitled to an estate for himself and heirs for ever, upon the decease of A, if A should happen to die before he attains to twenty-one; then the value of a life of twenty-one will be 12.9, which taken from 20, the perpetuity leaves 7.1, this multiply'd by $\frac{455}{541}$, the probability of a person of the age of 8 living to 21, gives 5.97, which multiply'd by 0.53, the present value of 1 L to be received *n* years hence, will be 3.16, this added to 14.3, the value of the life A, gives 17.46, which
sum

sum, deducted from the perpetuity, leaves 2.54, for the value of B's expectation, which is little more than $2\frac{1}{2}$ year's purchase.

PROBLEM XXIV.

Q expects an estate for himself and heirs for ever, after the extinction of a given number of lives A, B, C, provided this happens in a given number (n) of years; to find at how much the value of Q's expectation is to be estimated.

SOLUTION.

Let every thing be as in the preceding problem, only let P here denote the probability that all the lives A, B, C, &c. drop in n years; then by following the very same method of reasoning, there laid down, it will appear that $p m E$

$$+ \frac{1-a \times 1-b \times 1-c}{r}, \&c. + \frac{1-a \times 1-b \times 1-c}{r^2}, \&c.,$$

$$+ \frac{1-a \times 1-b \times 1-c}{r^n}, \&c. \text{ will be the true value}$$

lue

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lue of B's expectation; where the series is to be continued to n terms.

Note, the foregoing series, in case of two or three lives, might be referred to the tables, as that in the last Problem was, but the Advantage gained thereby, in computation, would not be so considerable.

O F



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S U C C E S S I V E L I V E S.

P R O B L E M X X V.

*S*upposing A to enjoy an annuity for life, and at his decease to have the nomination of a Successor, who is likewise to enjoy the annuity for his life; to find the present value of the two Successive Lives.

S O L U T I O N.

Let the value of the life A, in possession, be denoted by P; and let the life to be put in nomination, at the decease of A, be such, that the value of an annuity granted thereon, may be equal to Q, and let D be the required value of the two lives, and E the value of the annuity for ever,

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ever, and let $r, a, a', a'', \&c.$ be as in the preceding propositions. Therefore, since the probability that the first life fails, or that the second comes into possession, the first year is $1-a$; and as the total value of what the second life will be intitled to at the happening of this event, is Q , the expectation on the second, upon the contingency of coming into possession the first year, will, it is evident, be $\overline{1-a} \times Q$. Moreover, because the probability that the second life comes into possession the second year is $a-a'$, the expectation on the whole value of the annuity, upon the probability of entering into possession the second year, (discount for one year being allow'd) will be $\frac{a-a'}{r} \times Q$. And after the same manner will the expectations, upon the 3d, 4th, 5th, &c. years, appear to be $\frac{a-a''}{r^2} \times Q, \frac{a-a'''}{r^3} \times Q, \frac{a-a''''}{r^4} \times Q, \&c.$ respectively; therefore the sum of all these, or $Q \times 1 + \frac{a}{r} + \frac{a'}{r^2} + \frac{a''}{r^3} + \frac{a'''}{r^4}, \&c.$ $- Q \times a + \frac{a}{r} + \frac{a'}{r^2} + \frac{a''}{r^3} + \frac{a'''}{r^4}, \&c.$ must be equal to $(D-P)$ the *present, total, value*

lue of the second life. But $\frac{a}{r} + \frac{a'}{r^2} + \frac{a''}{r^3}$,
 &c. is the value of the life A; where-
 fore, by writing P instead of $\frac{a}{r} + \frac{a'}{r^2}$, &c.
 we have $Q \times \overline{1 + P} - Q \times rP = D - P$, and
 consequently $D = P + Q - \overline{r - 1} \times P$
 $= P + Q - \frac{PQ}{E}$; whence it appears, that
 if from the sum of the values of the two
 single lives be taken their product, divided
 by the perpetuity, the remainder will be
 the value of the two lives in succession.
 Q. E. I.

Otherwise,

Since the value of the second life in suc-
 cession, to be received at the decease of A,
 is to the value of the reversion for ever,
 to be received at the same time, in the
 ratio of Q to E, the present value of the
 former of these will be to the present va-
 lue of the latter in the same ratio; but the
 present value of the latter, or of the re-
 version for ever after the life of A, is given
 equal to $E - P$; therefore the present va-
 lue of the former, will be equal to $\frac{Q}{E}$

$\times \overline{E - P} = Q \times \overline{1 - \frac{P}{E}}$, which being added
 to P, the present value of the life A, gives

N
P

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$P + Q - \frac{PQ}{E}$, the very same as was before determined.

EXAMPLE.

Let the present value of the life A, in possession, be equal to 11.2 years purchase, and let the life put in nomination at the decease of A, be worth 16.3 years purchase, and the value of an annuity for ever, equal to 25 years purchase; then the sum of the values of the two lives will be 27.5, and their product 182.56; this last divided by 25, the value of the perpetuity, gives 7.3, which being subtracted from 27.5, there will remain 20.2, for the value of the two Successive Lives.

PROBLEM XXVI.

Three lives A, B, C, being given in succession; to find their present value.

SOLUTION.

Let the values of the three lives, considered as independent of each other, be respectively

pectively denoted by P, Q, R , and let E be the value of an annuity for ever. Therefore, since the present value of the two first lives in succession (A and B) is, by the last Problem, equal to $P + Q - \frac{PQ}{E}$, the value of the reversion of the annuity for ever, after these two lives, will be $E - P - Q + \frac{PQ}{E}$; and therefore, from the same method of reasoning as in the preceding Problem, it will be as $E : R :: E - P - Q + \frac{PQ}{E}$ to $R - \frac{PR - RQ}{E} + \frac{PQR}{EE}$, or $R \times 1 - \frac{P}{E} \times 1 - \frac{Q}{E}$, the present value of the third life; this therefore added to $P + Q - \frac{PQ}{E}$, the value of the two first, gives $P + Q + R - \frac{PQ - PR - QR}{E} + \frac{PQR}{EE} = E$ into $1 - 1 - \frac{P}{E} \times 1 - \frac{Q}{E} \times 1 - \frac{R}{E}$ for the required value of the three successive lives; from whence the value of any number of lives in succession, may be derived by inspection; and thence the following general Rule.

Multiply the value of each of the proposed lives by the interest of 1 L for one year, taking the several products from unity, and multiplying together all the remainders, let the product thus arising be

N 2

also

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also taken from unity, and the remainder multiply'd into the value of the annuity for ever; then will the result be the value of an annuity for all the successive lives.

EXAMPLE.

Let there be three lives given in succession, whose values separately consider'd, are respectively equal to 8, 10, and 15 years purchase, and let interest be at 4 *per cent.* then the several given values, multiply'd by 0.04, will be 0.32, 0.4, and 0.6; these severally subtracted from unity, leave 0.68, 0.6, and 0.4, whose product 0.1632 being taken from unity, there will remain 0.8368, and this multiply'd by 25, the perpetuity, gives 20.92 for the present value of the three lives.

PROBLEM XXVII.

Supposing A to purchase an estate, in copyhold, upon any number of lives P, Q, R, S, for the sum b, on condition that he and his heirs shall renew it continually, whenever any life becomes vacant, for the sum c; to find the present value of the whole purchase allow'd for that estate.

S O-

SOLUTION.

Let the values of the lives P, Q, R, &c. upon which the lease is first granted, be $p, q, r, \&c.$ and e the perpetuity, and let the values of the lives ($P', P'', P''', \&c.$) which follow in a direct succession from P, be denoted by $p', p'', p''', \&c.$ and those of the lives ($Q, Q', Q'', \&c.$) following in a direct succession from Q, by $q, q', q'', \&c.$ respectively, and so on, with regard to the rest of the lives R, S, T; then it will be as $p' : c :: p' \times 1 - \frac{p}{e}$, the present value of the life P' (by Prob. XXV.) to $c \times 1 - \frac{p}{e}$ the present value of the sum (c) to be paid at the decease of P, or the nomination of the new life P' ; and as $p'' : c :: p'' \times 1 - \frac{p}{e} \times 1 - \frac{p'}{e}$ the present value of the life P'' (by Prob. XXVI.) to $c \times 1 - \frac{p}{e} \times 1 - \frac{p'}{e}$ the present value of the sum (c) to be paid upon the decease of P' , or the nomination of P'' , the next life in this succession: And in the

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the same manner will the present values of the sums to be paid at the nomination of

the lives $\overset{m}{P}$, $\overset{m}{P}$, &c. be found $c \times 1 - \frac{p}{e}$

$\times 1 - \frac{p}{e} \times 1 - \frac{p}{e}$, and $c \times 1 - \frac{p}{e} \times 1 - \frac{p}{e}$

$\times 1 - \frac{p}{e} \times 1 - \frac{p}{e}$, &c. But the sum of

all these, or $c \times 1 - \frac{p}{e} + c \times 1 - \frac{p}{e}$

$\times 1 - \frac{p}{e} + c \times 1 - \frac{p}{e} \times 1 - \frac{p}{e} \times 1 - \frac{p}{e}$,

&c. continued *in infinitum*, is the present value of all the sums that may be paid from time to time, for the renewals of all the lives in this succession; and from the same way of reasoning, the present value of all the sums, that may be paid for the renewals of all the lives in the successions

$\overset{q}{Q}$, $\overset{q}{Q}$, $\overset{q}{Q}$, &c. $\overset{r}{R}$, $\overset{r}{R}$, $\overset{r}{R}$, &c. &c. will

appear to be $c \times 1 - \frac{q}{e} + c \times 1 - \frac{q}{e} \times 1 - \frac{q}{e}$

$+ c \times 1 - \frac{q}{e} \times 1 - \frac{q}{e} \times 1 - \frac{q}{e}$, &c. and

$c \times 1 - \frac{r}{e} + c \times 1 - \frac{r}{e} \times 1 - \frac{r}{e} + c \times 1 - \frac{r}{e}$

$\times 1 - \frac{r}{e} \times 1 - \frac{r}{e}$, &c. &c. respectively.

There-

Therefore, if to the sum of all these, the value b , paid at entring be added, the ag-

$$\begin{aligned} & \text{gregate, } b + c \times \overline{1 - \frac{p}{e}} + \overline{1 - \frac{p}{e}} \times \overline{1 - \frac{p}{e}} \\ & + \overline{1 - \frac{p}{e}} \times \overline{1 - \frac{p}{e}} \times \overline{1 - \frac{p}{e}}, \text{ \&c. } + \\ & c \times \overline{1 - \frac{q}{e}} + \overline{1 - \frac{q}{e}} \times \overline{1 - \frac{q}{e}} + \overline{1 - \frac{q}{e}} \\ & \times \overline{1 - \frac{q}{e}} \times \overline{1 - \frac{q}{e}}, \text{ \&c. } + c \times \overline{1 - \frac{r}{e}} \\ & + \overline{1 - \frac{r}{e}} \times \overline{1 - \frac{r}{e}} + \overline{1 - \frac{r}{e}} \times \overline{1 - \frac{r}{e}} \\ & \times \overline{1 - \frac{r}{e}}, \text{ \&c. \&c. will, it is evident, be the} \end{aligned}$$

present total value of all that A and his heirs may pay for the enjoyment of the estate for ever. Q. E. I.

COROLLARY I.

Hence, if the lives with which the lease is filled up from time to time, be equal to one another, the general expression will become $b + c \times \overline{1 - \frac{p}{e}} \times$

$$\overline{1 + \overline{1 - \frac{p}{e}}^2 + \overline{1 - \frac{p}{e}}^2 + \overline{1 - \frac{p}{e}}^3, \text{ \&c.}}$$

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$$+c \times 1 - \frac{q}{e} \times 1 + 1 - \frac{p}{e} + 1 - \frac{p}{e}, \&c.$$

which therefore, because $1 + 1 + \frac{p}{e}$
 $+ 1 - \frac{p}{e} + 1 - \frac{p}{e} + 1 - \frac{p}{e}, \&c.$ is e-
 qual to $\frac{e}{p}$, will become $b + \frac{c}{p} \times \overline{e-p}$
 $+ e - q + e - r, \&c.$

C O R O L L A R Y II.

But if all the lives, as well those upon
 which the lease is first granted, as those
 put into nomination afterwards, be equal to
 each other, the value allowed for the whole
 purchase will be $b + \frac{ec \times \overline{e-p}}{p}$.

C O R O L L A R Y III.

Lastly, if $b + c \times 1 - \frac{p}{e} + 1 - \frac{p}{e} \times 1 - \frac{p}{e},$

$\&c. + c \times 1 - \frac{q}{e} + 1 - \frac{q}{e} \times 1 - \frac{q}{e}, \&c.,$
 $\&c.$ be taken $= e$, we shall have $c =$

$$\frac{c-b}{1-\frac{p}{e} + 1-\frac{p}{e} \times 1-\frac{p}{e}, \&c. + 1-\frac{q}{e} + 1-\frac{q}{e} \times 1-\frac{q}{e}}$$

&c. shewing the true value that A and his heirs ought in justice to pay at each renewal; which value, when the lives renew'd with are all equal, will therefore be

$$\frac{c-b \times p}{e-p + e-q + e-r + e-s, \&c.}$$

The above Corollaries, expressed in words at length, afford the following practical Rules.

I. Subtract the sum of the values of all the lives, upon which the lease is first granted, from the perpetuity multiply'd by the number of those lives, and divide the remainder by the value of one of the equal lives, with which the lease is from time to time to be renewed, and multiply the quotient by the sum agreed upon to be paid for renewing, and the product will be the present value of all the sums that may be paid for all the renewals for ever; which added to the value paid at first entring will give the total value of the purchase.

II. Multiply the rent of one year by the present value of all the sums that may be paid for renewals (found as above) and divide the product by the perpetuity, and the quotient will be the sum by which the

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rent-roll of the first proprietor's estate, ought to be increased upon account of such renewals.

III. Take the difference between the value paid upon first entering, and the perpetuity, and multiply it by the value of one of the equal lives, with which the lease is to be constantly renewed, divide the product by the excess of the rectangle of the perpetuity into the given number of lives, above the sum of the values of all those lives, and the quotient will be the sum, which, in justice, ought to be constantly paid for renewing.

E X A M P L E.

Let the proposed estate be 100*l. per annum*, and the number of lives upon which the lease is granted be 3, and let their values, separately consider'd, be worth 10, 12, and 15 years purchase; let the sum paid upon entring be 1600*l.* and that for renewing 400*l.* and interest at 4 *per cent.* and suppose the purchaser to have the liberty of renewing with lives of what ages he shall think proper, or most to his own advantage.

Then the sum of the values of the three lives will be 37 years purchase, which taken from 75, three times the perpetuity,
leaves

leaves 38 years purchase; but the greatest value of an annuity for one single life (at 4 *per cent.*) is, by the table 16.4; wherefore, dividing 38 by 16.4, we have 2.317, which multiply'd by 4, the number of years purchase paid for each renewal, gives 9.268 years purchase, or 926*l.* 16*s.* for the present value of all the sums that may be paid from time to time for renewing; therefore 2526*l.* 16*s.* is the whole value of the purchase; from whence it will be found that the sum by which the rent-roll of the estate of the first proprietor, ought to be increased upon account of those renewals, is 37*l.* 1*s.* and that 388*l.* 6*s.* is the sum that ought to be paid, in justice, at each renewal.



O F
REVERSIONS,

WHERE THE

Expectation depends upon the Probability of one particular Life, in possession, surviving the rest.

LEMMA II.

THE ages of two persons *A* and *B* being given; to determine from the table of Observations, the probability which each of them has to survive the other.

Let *QO* represent the whole extent of life, continued from the instant of birth at *Q*, to the extremity of old age at *O*, and let *QA*, and *QB*, be the given ages of *A* and *B*, and *Q'A* and *Q'B* any other two corresponding ages to which they have a chance of

of attaining; and let $PbaO$ be a curve, whose ordinates Bb , $B'b$, Aa , &c. represent the numbers of persons answering, in the table of observations, to those ages; suppose cq as near as possible to $B'b$, and $A'm$ equal to Bc , and qz parallel to QO : put $Aa=a$, $A'a=a'$, $Bb=b$, $bz=b'$, and $qz=A'm=x$; and suppose A , if he be the survivor, to receive the sum S . Then will the probability of his receiving that sum, during the interval $A'm$, be compounded of the probability of his attaining to the age QA , and that of B 's dying in the corresponding interval Bc , of which, the former being $\frac{a'}{a}$, and the latter $\frac{b}{b'}$, the compound of both must consequently be $\frac{a'b}{ab'}$. Let this value, therefore, be defined (every where) by an area $A'mnFA$, supposing $EFnQ$ to be a curve, whose whole area $AOEA$ expresses the probability that A receives the sum S some time during the whole interval AO ; then by dividing $\frac{a'b}{ab'}$ by x ($=A'm$) we shall have $AF = \frac{a'b}{ab'x}$ for the general equation of this

curve;

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curve; from which, by a known method for approximating the areas of curves, by means of equidistant ordinates, may the area AOE A be determined; which area, as it expresses the probability that A lives to receive the sum S, must also express the probability that he out-lives B. Suppose, for instance, the age of A to be 40 years, and that of B 30 years, and suppose the interval A O (which may, with all the exactness here requisite, be confined to 50 years) to be divided into 5 equal parts at the points $\dot{A}$, g , b , i ; then will the successive values of $\dot{a}$, answering to these several ages QA , $Q\dot{A}$, Qg , Qb , &c. according to the table, be found 294, 204, 130, &c. and the decrements of life, against (30, 40, 50, 60, 70, and 80 years) the corresponding ages of B, equal to 9, 10, 8, 7, 5, and 3 respectively; therefore it will be, as 1 year, to the time $(Bc) \dot{x} ::$ so is 9, 10, 8, or 7, &c. the decrement in one year, to $9\dot{x}$, $10\dot{x}$, $8\dot{x}$, or $7\dot{x}$, &c. the decrement in the time $\dot{x}$ very near; which values, therefore, with those above, being successively wrote for $\dot{b}$ and $\dot{a}$, in the general expression $\frac{\dot{a}\dot{b}}{\dot{x}ab}$, the ordinates A E,

A F

A F, g G, &c. will, in this case, come out
 $\frac{294 \times 9}{294 \times 385}, \frac{204 \times 10}{294 \times 385}, \frac{130 \times 8}{294 \times 385}, \frac{69 \times 7}{294 \times 385}, \frac{29 \times 5}{294 \times 385}$
 and $\frac{0 \times 3}{294 \times 385}$ respectively: therefore the va-
 lue of the mean ordinate will here be
 $\frac{290299}{294 \times 385 \times 288} = 0.008905$, which multi-
 plied by A O = 50, gives 0.44525 for the
 required probability of A surviving B;
 whence the probability of the contrary,
 that B survives A, will be $1 - 0.44525 =$
 0.55475.

Note. That a mean, from a given num-
 ber of ordinates, may be had by the fol-
 lowing theorems, where $a, b, c, \&c.$ de-
 note the given ordinates ranged according
 to order, and m is the mean ordinate cor-
 responding.

$$3. \frac{a + 4b + c}{6} = m \quad 4. \frac{a + 3b + 3c + d}{8} = m$$

$$5. \frac{7a + 32b + 12c + 32d + 7e}{90} = m$$

$$6. \frac{19a + 75b + 50c + 50d + 75e + 19f}{288} = m$$

$$7. \frac{41a + 216b + 27c + 272d + 27e + 216f + 41g}{840} = m$$

PRO-

P R O B L E M X X V I I I .

A and B are in joint possession of an annuity; which, if A be the longer liver, is, after both lives are extinct, to belong to C and his heirs for ever: To find the present value of the expectation of C, on that annuity.

S O L U T I O N .

From the value of the annuity for ever, subtract the value of the two lives in possession, and multiply the remainder by the probability of A surviving B (found by the preceding Lemma) and the product will be the value required.

For since the value of the reversion which C and his heirs expect, did it not depend on A's surviving B, would be the excess of the value of the annuity for ever, above the value of the two lives A and B, that excess multiply'd by the probability of A surviving, must consequently be the true value of the reversion, allowing for that contingency.

EXAMPLE.

Let the age of A be 40, that of B 30, and interest of money at 4 per cent. then the value of the two lives A and B will be 15.8 years purchase (by Case II. Prob. X.) which taken from 25 years purchase, the value of the annuity for ever, there will remain 9.2, and this multiply'd by 0.445, the probability of A surviving B, gives 4.1 for the number of years purchase required.

PROBLEM XXIX.

There are three persons A, B, C, the two former of whom A, B, enjoy an annuity between them, which annuity, if A survives B, is afterwards to belong equally to A and C, during their joint continuance, and then intirely to C for life, if he be the last survivor; to find the value of C's expectation on that annuity.

SOLUTION.

Take the excess of the value of the three lives A, B, C, above the value of the two lives A, B, and also the excess of the value

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lue of the two *joint* lives A, C, above that of the three *joint* lives A, B, C; multiply the former of them by the probability that A survives B, and to the product add half the latter, and the sum thus arising will be the true value of C's expectation.

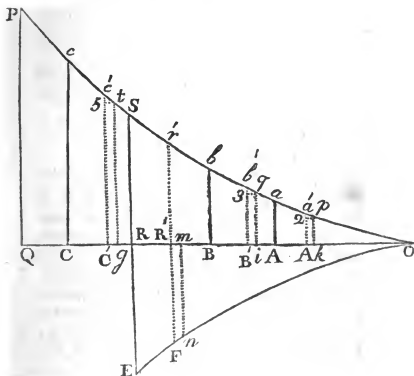
For since the first of the abovemention'd excesses would be the present value of all the rents that C might receive after the decease of A and B, was not that value to depend on A's surviving B, the true value of those rents, allowing for that contingency, must be the product of that excess, by the probability of A surviving B: therefore, as the latter of those excesses is (by Prob. V.) double the value of all the other rents that he may receive, during the joint continuance of himself and A, the solution is manifest.

LE M M A III.

The ages of any number of persons R, A, B, C, D, &c. being given; to find the probability that any one of them, pitch'd upon, as R, shall survive the rest.

Let QO represent the whole term of life, continued from the time of birth at Q, to the extremity of old age at O; and let QR, QA, QB, QC, &c. be the given ages of R, A, B, C, &c. respectively,

ly, and $Q\acute{K}$, $Q\acute{A}$, $Q\acute{B}$, $Q\acute{C}$, &c. any other contemporary ages to which those persons have a chance of living; let PSO be a curve whose ordinates Cc , RS , Bb , &c. every where express the numbers of persons, in the table of observations, answering to the corresponding ages QC , QR , QB , &c. let $k\acute{p}$ be as near as can be to $\acute{A}\acute{a}$, and $\acute{A}\acute{k}$, $\acute{B}\acute{i}$, $\acute{K}\acute{m}$, $\acute{C}\acute{g}$, &c. all equal to one another, and let $p\acute{2}$, $q\acute{3}$, and $t\acute{5}$, &c. be parallel to QO . Put RS



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$=r, Rr=r, Rm=x, Aa=a, A'a=a,$
 $a_2=a, Bb=b, B'b=b, b_3=b, Cc=c,$
 $C'c=c, c_5=c, \&c.$ and let R, in case he
 be the last survivor, be intitled to receive
 the sum S; then will the probability of his
 receiving that sum, in the interval Rm ,
 depend upon these events; first, that he
 attains to the age QR ; secondly, that
 only one of the other persons shall then
 remain alive; and lastly, that this person
 shall also go out of being, during that in-
 terval; wherefore, since the probability that
 all the lives, except R and A, are extinct before
 that time is $\frac{b-b'}{b} \times \frac{c-c'}{c} \times \frac{d-d'}{d}$, &c. and
 that of A dying in the said interval, equal
 to $\frac{a}{a}$, it is manifest that $\frac{r}{r} \times \frac{a}{a} \times \frac{b-b'}{b} \times$
 $\frac{c-c'}{c} \times \frac{d-d'}{d}$, &c. will be that part of the
 probability of R's receiving the sum S, in
 the foresaid interval, which depends upon
 the chance of A's being the last survivor,
 except R; and for the like reasons it will
 appear, that the other parts of that pro-
 bability, depending upon the chance of
 B, C, D, &c. being the last survivor but
 R, will be $\frac{r}{r} \times \frac{b}{b} \times \frac{a-a'}{a} \times \frac{c-c'}{c} \times \frac{d-d'}{d}$, &c.
 and

and $\frac{r}{r} \times \frac{c}{c} \times \frac{a-a'}{a} \times \frac{b-b'}{b} \times \frac{d-d'}{d}$, &c. &c.
respectively; therefore the sum of all these,

or its equal $p \times \frac{a}{a-a'} + \frac{b}{b-b'} + \frac{c}{c-c'} + \frac{d}{d-d'}$,

&c. by putting $p = \frac{r}{r} \times \frac{a-a'}{a} \times \frac{b-b'}{b} \times \frac{c-c'}{c}$,

&c. will be the whole probability of receiving the said sum during that interval. Let this probability be, every where, represented by the area $RmnFR$, supposing EFO to be a curve, whose whole area $REOR$ expresses the probability that R receives the sum S , sometime during the whole interval RO , or survives all the other persons A, B, C ; then by dividing

$p \times \frac{a}{a-a'} + \frac{b}{b-b'} + \frac{c}{c-c'} + \frac{d}{d-d'}$, &c. by x

($= Rm$) we shall have $RF = \frac{p}{x} \times$

$\frac{a}{a-a'} + \frac{b}{b-b'} + \frac{c}{c-c'} + \frac{d}{d-d'}$, &c. for the

general equation of this curve; from which, by the method and theorems laid down in Lemma II. the area $REOR$, and consequently the probability expressed thereby, may be determined.

COROLLARY.

If the probability of life be considered as decreasing uniformly, then will P S O become a right-line, and $a - a'$, $b - b'$, $c - c'$, &c. as also $\dot{a}$, $\dot{b}$, $\dot{c}$, &c. equal to each other; therefore, by putting $a - a' = x$, $\dot{a} = \dot{x}$, and n the whole number of persons given, the probability of receiving the sum S in the interval $R m$, will be defined by

$$\frac{n-1}{r a b c d, \&c.} \times r z^{n-2} z - z^{n-1} z; \text{ whose flu-}$$

ent (after proper correction) will, when $x=r$, be $\frac{1}{r a b c d, \&c.}$ multiply'd into the two following series $r a^{n-1} + r a \times b^{n-2} - a^{n-2}$
 $+ r a b \times c^{n-3} - b^{n-3} + r a b c \times d^{n-4} - c^{n-4}$
 $\&c. - \frac{n-1 \times a^n}{n} - \frac{n-2 \times a \times b^{n-1} - a^{n-1}}{n-1}$
 $- \frac{n-3 \times a b \times c^{n-2} - b^{n-2}}{n-2}$
 $- \frac{n-4 \times a b c \times d^{n-3} - c^{n-3}}{n-3}, \&c. \text{ where}$

each series is to be continued to as many terms, as will express the place which the person proposed obtains, reckoning according to seniority, provided that person be not the

the youngest of all, in which case, one term less than that number is to be taken. But the fluent thus found, is the required probability of survivorship, according to the above hypothesis, and, with a little correction, will serve sufficiently near in real cases.

PROBLEM XXX.

To determine the value of a reversion for ever, after any number of lives A, B, C, &c. upon the contingency of one particular life A being the last survivor.

SOLUTION.

From the value of the annuity for ever subtract the value of all the given lives, and multiply the remainder by the probability that A is the last survivor (found by the above Lemma) and the product will be the value required; which will appear manifest from the reasons already laid down in the two preceding problems.

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A TABLE shewing the present value of one pound, to be received at the end of any number of years, not exceeding 90, discounting at the rates of 5, 4, and 3 per cent. compound interest.

Years.	Value at 5 per cent.	Value at 4 per cent.	Value at 4 per cent.	Years.	Value at 5 per cent.	Value at 4 per cent.	Value at 3 per cent.
1	.9524	.9615	.9709	21	.3589	.4388	.5375
2	.9070	.9245	.9426	22	.3418	.4219	.5219
3	.8638	.8890	.9151	23	.3255	.4057	.5067
4	.8227	.8548	.8885	24	.3100	.3901	.4919
5	.7835	.8219	.8626	25	.2953	.3757	.4776
6	.7462	.7903	.8375	26	.2812	.3607	.4637
7	.7107	.7599	.8131	27	.2678	.3468	.4502
8	.6768	.7307	.7894	28	.2551	.3335	.4371
9	.6446	.7026	.7664	29	.2429	.3206	.4243
10	.6139	.6756	.7441	30	.2314	.3083	.4120
11	.5847	.6496	.7224	31	.2204	.2965	.4000
12	.5568	.6246	.7014	32	.2099	.2851	.3883
13	.5303	.6006	.6809	33	.1999	.2741	.3770
14	.5051	.5775	.6611	34	.1903	.2636	.3660
15	.4810	.5553	.6419	35	.1813	.2534	.3554
16	.4581	.5339	.6232	36	.1726	.2437	.3450
17	.4363	.5134	.6050	37	.1644	.2343	.3350
18	.4155	.4936	.5874	38	.1566	.2253	.3252
19	.3957	.4746	.5703	39	.1491	.2166	.3158
20	.3769	.4564	.5537	40	.1420	.2083	.3066

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Years.	Value at 5 per cent.	Value at 4 per cent.	Value at 3 per cent.	Years.	Value at 5 per cent.	Value at 4 per cent.	Value at 3 per cent.
41	.1353	.2003	.2977	66	.0399	.0751	.1421
42	.1288	.1926	.2890	67	.0380	.0722	.1380
43	.1227	.1852	.2805	68	.0362	.0695	.1340
44	.1169	.1780	.2724	69	.0345	.0668	.1301
45	.1113	.1712	.2644	70	.0329	.0642	.1263
46	.1060	.1646	.2567	71	.0313	.0617	.1226
47	.1010	.1583	.2493	72	.0298	.0594	.1190
48	.0961	.1522	.2420	73	.0284	.0571	.1156
49	.0916	.1463	.2349	74	.0270	.0549	.1122
50	.0872	.1407	.2281	75	.0257	.0528	.1089
51	.0830	.1353	.2215	76	.0245	.0508	.1058
52	.0791	.1301	.2150	77	.0233	.0488	.1027
53	.0753	.1251	.2087	78	.0222	.0469	.0997
54	.0717	.1203	.2027	79	.0212	.0451	.0968
55	.0683	.1156	.1968	80	.0202	.0434	.0940
56	.0651	.1112	.1910	81	.0192	.0417	.0912
57	.0620	.1069	.1855	82	.0183	.0401	.0886
58	.0590	.1028	.1801	83	.0174	.0386	.0860
59	.0562	.0989	.1748	84	.0166	.0371	.0835
60	.0535	.0951	.1697	85	.0158	.0357	.0811
61	.0510	.0914	.1648	86	.0151	.0343	.0787
62	.0485	.0879	.1600	87	.0143	.0330	.0764
63	.0462	.0845	.1553	88	.0136	.0317	.0742
64	.0440	.0813	.1508	89	.0130	.0305	.0720
65	.0419	.0781	.1464	90	.0124	.0293	.0699

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A TABLE shewing the present value of an annuity of one pound for any number of years, not exceeding 90, when interest is at 5, 4, and 3 per cent.

Years.	Value at 5 per cent.	Value at 4 per cent.	Value at 3 per cent.	Years.	Value at 5 per cent.	Value at 4 per cent.	Value at 3 per cent.
1	0.952	0.961	0.971	21	12.821	14.029	15.415
2	1.859	1.886	1.913	22	13.163	14.451	15.939
3	2.723	2.775	2.829	23	13.488	14.857	16.444
4	3.546	3.630	3.717	24	13.799	15.247	16.936
5	4.329	4.452	4.580	25	14.094	15.622	17.413
6	5.076	5.242	5.497	26	14.375	15.983	17.877
7	5.786	6.002	6.230	27	14.643	16.329	18.327
8	6.463	6.733	7.020	28	14.898	16.663	18.764
9	7.108	7.435	7.786	29	15.141	16.984	19.188
10	7.721	8.111	8.530	30	15.372	17.292	19.600
11	8.306	8.760	9.253	31	15.593	17.588	20.000
12	8.863	9.385	9.954	32	15.803	17.873	20.389
13	9.393	9.985	10.635	33	16.002	18.148	20.766
14	9.899	10.561	11.296	34	16.193	18.411	21.132
15	10.381	11.118	11.938	35	16.374	18.665	21.487
16	10.838	11.652	12.561	36	16.547	18.908	21.832
17	11.274	12.161	13.166	37	16.711	19.142	22.167
18	11.689	12.651	13.753	38	16.868	19.368	22.492
19	12.085	13.134	14.324	39	17.017	19.584	22.808
20	12.462	13.590	14.877	40	17.159	19.793	23.115

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Years.	Value at 5 per cent.	Value at 4 per cent.	Value at 3 per cent.	Years.	Value at 5 per cent.	Value at 4 per cent.	Value at 3 per cent.
41	17.294	19.993	23.412	66	19.201	23.122	28.595
42	17.423	20.186	23.701	67	19.239	23.194	28.733
43	17.546	20.371	23.982	68	19.275	23.263	28.867
44	17.663	20.549	24.254	69	19.310	23.330	28.997
45	17.774	20.720	24.519	70	19.343	23.394	29.123
46	17.880	20.885	24.775	71	19.374	23.456	29.246
47	17.981	21.043	25.025	72	19.404	23.516	29.365
48	18.077	21.195	25.267	73	19.432	23.573	29.481
49	18.169	21.341	25.502	74	19.459	23.628	29.593
50	18.256	21.482	25.730	75	19.485	23.680	29.702
51	18.339	21.617	25.951	76	19.509	23.731	29.808
52	18.418	21.747	26.166	77	19.533	23.780	29.910
53	18.493	21.873	26.375	78	19.555	23.827	30.010
54	18.565	21.993	26.578	79	19.576	23.872	30.108
55	18.633	22.109	26.774	80	19.596	23.915	30.201
56	18.698	22.220	26.965	81	19.615	23.957	30.292
57	18.760	22.327	27.151	82	19.634	23.997	30.381
58	18.819	22.430	27.331	83	19.652	24.036	30.467
59	18.876	22.528	27.506	84	19.668	24.073	30.550
60	18.929	22.623	27.676	85	19.684	24.108	30.631
61	18.980	22.715	27.840	86	19.699	24.143	30.710
62	19.029	22.803	28.000	87	19.713	24.176	30.786
63	19.075	22.887	28.156	88	19.727	24.207	30.860
64	19.119	22.968	28.306	89	19.740	24.238	30.932
65	19.161	23.046	28.453	90	19.752	24.267	31.002

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A TABLE shewing the difference of values of an annuity for life, at the rates of $3\frac{1}{2}$, 4 and $4\frac{1}{2}$ per cent. interest.

Value of an Annuity at 4 per cent.	Diff. of values at 4 and $4\frac{1}{2}$ per cent.	Diff. of values at 4 and $3\frac{1}{2}$ per cent.
6	0.2	0.2
7	0.2	0.2
8	0.3	0.3
9	0.4	0.5
10	0.5	0.6
11	0.6	0.7
12	0.7	0.8
13	0.8	0.9
14	0.9	1.0
15	1.0	1.2
16	1.1	1.3

This table may be used as a supplement to that in page 38, &c. thus; Suppose it were required to find the value of a life of forty-four, at the rate of $4\frac{1}{2}$ per cent. interest; then looking in tab. I. page 39, against 44 years, under 4 per cent. you will find 11.0, with this enter the first column of

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of the last table, and against it, in the second column, you will have 0.6, which subtracted from 11.0, leaves 10.4 for the value required.

A METHOD for investigating the values of annuities upon lives, by approximation, without the help of tables.

Because there may be occasion sometimes to know the values of lives computed at different rates of interest, from any exhibited in the foregoing tables, and as the general method for this purpose, laid down in the former part of this book, is too tedious for common practice, I have endeavoured to remove that inconveniency by help of some easy approximations. To effect this, I found it necessary to consider the values of lives in two different cases, one when the given age is less than 45 years, and the other when it is greater.

R U L E I.

To find the value of an assigned life, whose age is not less than 45 years.

Multiply the difference between the given age and 92 years, by the interest of 1 l. for one year, add 2.47 to the product, and
divide

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divide the said difference by the product so increased, then the quotient will be the answer.

Note. This rule comes so near the truth, that the error for the general part does not amount to $\frac{1}{10}$, and scarce ever to $\frac{2}{10}$ of a year's purchase.

EXAMPLE I.

Let the proposed age be fifty years, and interest at 10 *per cent.* then subtracting 50 from 92, we have 42, which multiply'd by 0.1, the interest of 1 *l.* will be 4.2, and this added to 2.47, gives 6.67; by which divide 42, and there will come out 6.3 for the value required.

EXAMPLE II.

Suppose the given age to be 55 years, and interest at 4 *per cent.* Here the difference between the given age and 92 years, will be 37, and the interest of 1 *l.* equal to 0.04; with which proceed as in the last example, and the required value will be found 9.36; but if the rate of interest had been $4\frac{1}{2}$ *per cent.* the value then had been only 8.94; the like of any other.

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I would willingly have given a rule similar to the foregoing, for the other case where the given age is less than 45, but have not been able to find out any thing either so general or so simple, as that approximation for older lives; however, if the rate of interest be not less than 3, nor greater than 10 *per cent.* nor the age propounded, less than 12 years, the value of such a life may be always had by the following rule, to a degree of exactness equal to the former.

R U L E II.

To find the value of any assigned life, whose age is neither less than 12, nor greater than 45 years.

Take the difference between the given age and 29 years, also between the same age and 100 years, and let the former of those differences be multiply'd by 10 times the interest of 1 *l.* for one year, and the product added to, or subtracted from the latter, according as the given age is greater or less than 29 years, and let the sum or remainder be multiply'd by 40, and the product be reserved: add 100 years to the given age, and multiply the sum by 22 times the interest of 1 *l.* for one year; add 100 to the product, and divide the reserved

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reserved product by this sum, and the quotient will be the value required.

EXAMPLE.

Let it be proposed to find the value of a life of twenty, computed at the rate of 5 per cent.

Here the first mentioned difference will be 9, and the other 80, and therefore the former multiply'd by 0.5, or 10 times the interest of 1%. will be 4.5, which subtracted from 80, leaves 75.5, and this multiply'd by 40, will give 3020, for the product to be reserved. Moreover the given age, increased by 100 years, is 120, which multiply'd by 1.1, ($= 22 \times .05$) will be 132; therefore dividing 3020 by 232 ($= 100 + 132$) we have 13.1 for the number of years purchase required.

It will be needless (I presume) to offer any thing farther by way of example; but, for the reader's satisfaction, and to remove any doubt that he may entertain with regard to the exactness of the above methods of solution, I have thought proper to add the subsequent table; which shews the values of lives to every 10th year of age, both according to those methods, and also according to the table of observations.

Age

Age.	Value at 5 per ct. by the rules.	Value at 5 per cent. accord. to obs.	Value at 4 per ct. by the rules.	Value at 4 per cent. accord. to obs.	Value at 3 per ct. by the rules.	Value at 3 per cent. accord. to obs.
20	13.01	13.0	14.86	14.8	17.25	17.2
30	11.60	11.6	13.13	13.1	15.11	15.0
40	10.31	10.3	11.53	11.5	13.14	13.2
50	9.19	9.2	10.12	10.1	11.26	11.4
60	7.86	7.9	8.53	8.4	9.36	9.2
70	6.13	6.2	6.56	6.5	7.02	6.9

The two foregoing rules are accommodated to the same observations as the preceding tables, as the best, undoubtedly, for the city of *London*, and parts adjacent; but if any one be desirous of an approximation according to the *Breslau* observations, the following rule, in any case where the proposed age is not less than 30 years, nor the rate of interest less than 3, nor greater than 6 per cent. will give the true answer very nearly. The rule is this,

Multiply the difference between the given age and 87 years, by $\frac{8}{10}$ of the interest of 1 l. for one year, and add 1.9 to the product, and divide the said difference made less by 2, by the product so increased, and then the result will be the value required.

R

Having

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Having now shewn how to approximate the values of single lives, it remains next to lay down something in relation to annuities upon two or three lives.

R U L E I I I.

To find the value of two lives A and B.

Multiply the value of the youngest life A by the interest of 1 *l.* for one year, and subtract the product, and also half the product, each from unity, dividing the last remainder by the former; multiply the value of the life A by the quotient thus arising, and divide $\frac{4}{10}$ of the square of the value of the oldest life B, by this product, then the quotient, added to the value of the life A, will give the required value of the longest of the two lives.

E X A M P L E.

Suppose the value of the youngest life A equal to 12 years purchase, and that of the oldest B, equal to 10 years purchase, reckoning interest of money at 4 *per cent.* then the value of the life A multiply'd by 0.04, the interest of 1 *l.* for one year, will be 0.48, and half thereof equal to 0.24; these severally taken from unity, leave 0.52 and 0.76, the former of which divided by the latter,

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latter, gives 1.46, and this multiply'd by the value of the life A, will be 17.52 ; therefore by dividing $\frac{4}{10}$ of the square of the value of the life B, by 17.52, we have 2.3, which added to 12, gives 14.3 for the value required.

R U L E I V.

To find the value of the three lives A, B, C.

First, find the excess of the value of the two youngest lives A, B, above that of the youngest life A, by Rule III. then divide the value of the oldest life C, by the value of the life B, and cube the quotient, and multiply that cube by half the said excess ; then the product added to the value of the two lives A and B, will be the value required.

E X A M P L E.

Let the value of the life A be supposed equal to 12 years purchase, that of B equal to 10 years purchase, and that of C equal to 8 years purchase, and interest at 4 *per cent.* then the excess of the value of the two lives A, B, above that of the youngest life A will be 2.3 (by Rule III.) moreover, the value of the life C being divided by

R 2

that

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that of B, we have 0.8, the cube whereof is 0.512, which multiply'd by 1.15, half the abovesaid excess, gives 0.588, or 0.6 nearly, and this added to 14.3, the value of the two lives A, B, gives 14.9, for the value of all the three lives.

Note. The two last rules will serve indifferently, either according to the *London*, or according to the *Breslau* observations, the error, in either case, seldom exceeding $\frac{1}{8}$ of a year's purchase, as I have found by many repeated trials.

I shall conclude this little tract with the Solution of the following Problem, which tho' it relates not immediately to the subject of annuities, depends nevertheless upon the same principles.

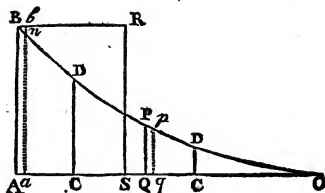
P R O B L E M XXXI.

To determine from a table of observations, and the bills of mortality of any place, the number of souls contained in that place.

S O L U T I O N.

Let A O represent the utmost extent of life continued from the time of birth at A, to the extremity of old age at O, and let B D P O be a curve, whose ordinates C D, Q P, &c. are, every where, to one another,

as the number of persons (found in the table) that arrive to the corresponding ages $A C$, $A Q$, &c. let ab be taken very near, and parallel to AB , and qp at the same distance from $Q P$, and let the rectangle $B S$ represent the number of persons which die in the proposed place, in any given interval of time $A S$, as found from the bills of mortality ; then will the number of persons that



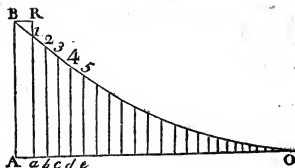
come into, or go out of, being in the interval Aa , be represented by the rectangle Ba ; and consequently the number of all the living, whose ages are comprized in this interval, by the area $B A a n B$, or the same rectangle Ba very near : Wherefore, since the number of individuals, whose ages now take up Aa , will in the time $A Q$ be reduced in the ratio of $A B$ to $P Q$, so as to be expounded by the area $P Q q p P$, the number of all the living at the end of that time,

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time, whose ages are then included between Q and q , will be represented by the area $PQq p P$; since it is evident that the ages of all who come into being, before or after those now included between A and a , will then be either greater than $A q$, or less than $A Q$. Therefore as the number of the living, at all equal ages, is supposed to continue constantly the same, and as the above reasoning holds every where throughout the whole extent of life $A O$, it is manifest that the number of all the living, at any one time, will be expounded by the whole curvilinear area $A O P B A$, and therefore will be to the number of persons dying, in the proposed interval of time $A S$, as that area to the rectangle BS . Q. E. I.

C O R O L L A R Y.

Hence if the whole term of life $A O$ be divided into several small intervals $A a, a b, b c, \&c.$ so that each interval may represent one whole year, and ordinates $A B, a 1, b 2, \&c.$ be described at the points of division, and the first $A B$ be taken 1280, then will $a 1, b 2, c 3, \&c.$ according to the foregoing table of observations, taken from the bills of mortality of the city of *London*, be 870, 700, 635, 600, $\&c.$ respectively; from whence the area $A B O A$

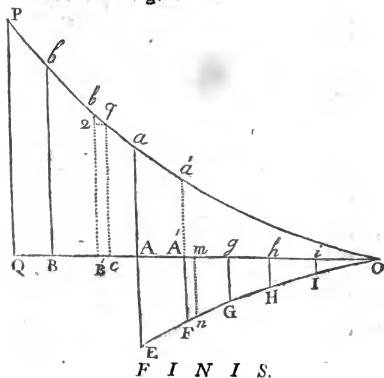


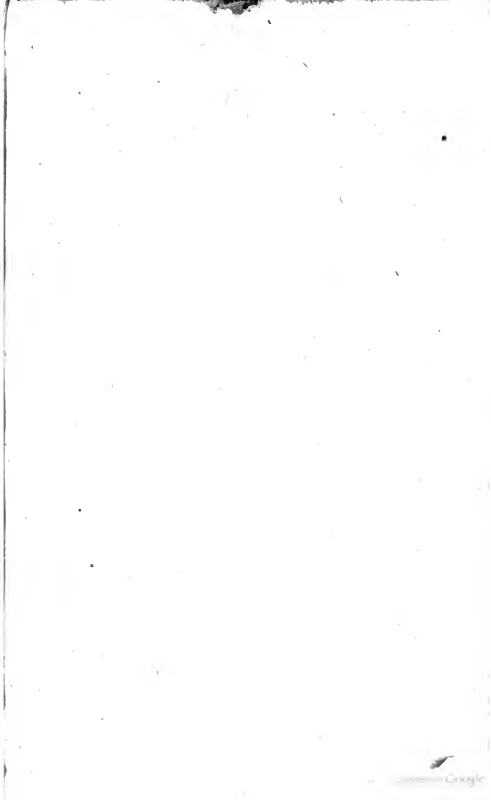
will come out 25500 very near, which area is, therefore, to the rectangle Ba, as 25500 to 1280, or as 20 to 1 nearly; whence it appears that the number of the living, at any one time, born within the bills of mortality of this city, is to the number of births happening yearly within the same bills (taken at a medium) as 20 to 1, very near; but since the number of burials happening yearly, always exceeds the number of births, by reason (as has been before observed) of the continual afflux of people from all parts to town, the proportion of that number to the whole body of the inhabitants, will be considerably different from the proportion above given; yet may be nearly estimated by comparing together the number of burials and christenings, &c.

And (from a method too tedious to be inserted here) I make it as 1 to 26 very near. I know indeed that a certain author, considerable in these kind of disquisitions, give,

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gives the proportion very different from that above, *viz.* as 1 to 35; but this, I apprehend could only be owing to a want of observations; because, if we suppose as many persons to live, at any one time, as die in 35 years, then it will follow, that the expectation or share of life due to each infant, at its coming into the world, taken at a medium, is 35 years; but it evidently appears from the foregoing table, and other undoubted observations, that there is scarce any part of life wherein the expectation is so great as 35 years, much less in the very beginning of it, attended by so many casualties and dangers.







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